BIG ARITHMETIC DIVISORS ON THE PROJECTIVE SPACES OVER $\mathbb{Z}$

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CONTENTS

INTRODUCTION

Let $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1, \ldots, T_n])$, $H_i = \{T_i = 0\}$ and $z_i = T_i/T_0$ for $i = 0, 1, \ldots, n$. Let us fix a sequence $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$ of positive numbers. We define a H_0-Green function $g_{\boldsymbol{a}}$ of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^n(\mathbb{C})$ and an arithmetic divisor $\overline{D}_{\boldsymbol{a}}$ of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^n_{\mathbb{Z}}$ to be

$$g_{\boldsymbol{a}} := \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2) \quad \text{and} \quad \overline{D}_{\boldsymbol{a}} := (H_0, g_{\boldsymbol{a}}).$$

In this paper, we will observe several properties of $\overline{D}_{\boldsymbol{a}}$ and give the exact form of the Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$. Further, we will show that, if $n \geq 2$ and $\overline{D}_{\boldsymbol{a}}$ is big and not nef, then, for any birational morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth and normal arithmetic varieties, we can not expect a suitable Zariski decomposition of $f^*(\overline{D}_{\boldsymbol{a}})$. In this sense, the results in [9] are nothing short of miraculous, and arithmetic linear series are very complicated and have richer structure than what we expected. We also give a concrete construction of Fujita's approximation of $\overline{D}_{\boldsymbol{a}}$. The following is a list of the main results of this paper.

Main Results. *Let $\varphi_{\boldsymbol{a}} : \mathbb{R}^{n+1}_{\geq 0} \to \mathbb{R}$ be a function given by*

$$\varphi_{\boldsymbol{a}}(x_0, x_1, \ldots, x_n) := -\sum_{i=0}^{n} x_i \log x_i + \sum_{i=0}^{n} x_i \log a_i,$$

and let

$$\Theta_{\boldsymbol{a}} := \{(x_1, \ldots, x_n) \in \Delta_n \mid \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \geq 0\},$$

where $\Delta_n := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\right\}$. Then the following properties hold for $\overline{D}_{\boldsymbol{a}}$:

(1) *$\overline{D}_{\boldsymbol{a}}$ is ample if and only if $a_0 > 1, a_1 > 1, \ldots, a_n > 1$.*

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(2) $\overline{D}_{\boldsymbol{a}}$ *is nef if and only if* $a_0 \geq 1, a_1 \geq 1, \ldots, a_n \geq 1$.
(3) $\overline{D}_{\boldsymbol{a}}$ *is big if and only if* $a_0 + a_1 + \cdots + a_n > 1$.
(4) $\overline{D}_{\boldsymbol{a}}$ *is pseudo-effective if and only if* $a_0 + a_1 + \cdots + a_n \geq 1$.

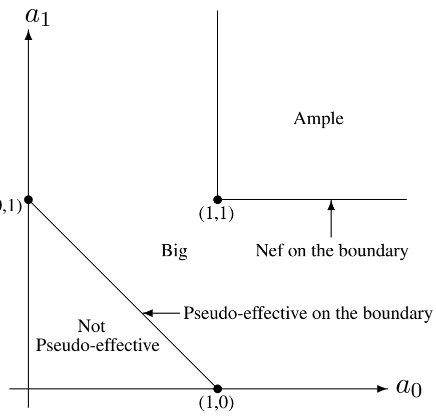

FIGURE 1. Geography of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$

(5) $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) \neq \{0\}$ *if and only if* $l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n \neq \emptyset$. *As consequences, we have the following:*
(5.1) *We assume that* $a_0 + a_1 + \cdots + a_n = 1$. *For a positive integer* l,

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) = \begin{cases} \{0, \pm z_1^{la_1} \cdots z_n^{la_n}\} & \text{if } la_1, \ldots, la_n \in \mathbb{Z}, \\ \{0\} & \text{otherwise.} \end{cases}$$

In particular, if $\boldsymbol{a} \notin \mathbb{Q}^{n+1}$, *then* $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) = \{0\}$ *for all* $l \geq 1$.
(5.2) *For any positive integer* l, *there exists* $\boldsymbol{a} \in \mathbb{Q}^{n+1}_{>0}$ *such that* $\overline{D}_{\boldsymbol{a}}$ *is big and*

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, k\overline{D}_{\boldsymbol{a}}) = \{0\}$$

for all k *with* $1 \leq k \leq l$.
(6) $\left\langle \hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) \right\rangle_{\mathbb{Z}} = \bigoplus_{(e_1,\ldots,e_n) \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \mathbb{Z} z_1^{e_1} \cdots z_n^{e_n}$ *if* $l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n \neq \emptyset$.
(7) (*Integral formula*) *The following formulae hold:*

$$\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = \frac{(n+1)!}{2} \int_{\Theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) dx_1 \cdots dx_n,$$

and

$$\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \frac{(n+1)!}{2} \int_{\Delta_n} \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

In particular, $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}})$ *if and only if* $\overline{D}_{\boldsymbol{a}}$ *is nef.*
(8) (*Zariski decomposition for* $n = 1$) *We assume* $n = 1$. *The Zariski decomposition of* $\overline{D}_{\boldsymbol{a}}$ *exists if and only if* $a_0 + a_1 \geq 1$. *Moreover, the positive part of* $\overline{D}_{\boldsymbol{a}}$ *is given by* $(\theta_{\boldsymbol{a}} H_0 - \vartheta_{\boldsymbol{a}} H_1, p_{\boldsymbol{a}})$, *where* $\vartheta_{\boldsymbol{a}} = \inf \Theta_{\boldsymbol{a}}$, $\theta_{\boldsymbol{a}} = \sup \Theta_{\boldsymbol{a}}$ *and*

$$p_{\boldsymbol{a}}(z_1) = \begin{cases} \vartheta_{\boldsymbol{a}} \log |z_1|^2 & \text{if } |z_1| < \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}, \\ \log(a_0 + a_1|z_1|^2) & \text{if } \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \leq |z_1| \leq \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ \theta_{\boldsymbol{a}} \log |z_1|^2 & \text{if } |z_1| > \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \end{cases}$$

In particular, if $a_0 + a_1 = 1$, then the positive part is $-a_1\widehat{(z_1)}$.

(9) (*Impossibility of Zariski decomposition for $n \geq 2$*) *We assume $n \geq 2$. If $\overline{D}_{\boldsymbol{a}}$ is big and not nef (i.e., $a_0 + \cdots + a_n > 1$ and $a_i < 1$ for some i), then, for any birational morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth and normal arithmetic varieties, there is no decomposition $f^*(\overline{D}_{\boldsymbol{a}}) = \overline{P}+\overline{N}$ with the following properties:*

(9.1) *$\overline{P}$ is a nef and big arithmetic $\mathbb{R}$-divisor of $(C^0 \cap \mathrm{PSH})$-type on X.*

(9.2) *$\overline{N}$ is an effective arithmetic $\mathbb{R}$-divisor of C^0-type on X.*

(9.3) *For any horizontal prime divisor Γ on X (i.e. Γ is a reduced and irreducible divisor on X such that Γ is flat over $\mathbb{Z}$),*

$$\begin{aligned}&\mathrm{mult}_\Gamma(N)\\ &\qquad \leq \inf\left\{\mathrm{mult}_\Gamma(f^*(H_0) + (1/l)(\phi)) \mid l \in \mathbb{Z}_{>0},\ \phi \in \hat{H}^0(lf^*(\overline{D}_{\boldsymbol{a}})) \setminus \{0\}\right\}.\end{aligned}$$

(10) (*Fujita's approximation*) *We assume that $\overline{D}_{\boldsymbol{a}}$ is big. Let $\mathrm{Int}(\Theta_{\boldsymbol{a}})$ be the set of interior points of $\Theta_{\boldsymbol{a}}$. We choose $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in \mathrm{Int}(\Theta_{\boldsymbol{a}}) \cap \mathbb{Q}^n$ such that*

$$\frac{(n+1)!}{2}\int_\Theta \phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})d\boldsymbol{x} > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon,$$

where $\Theta := \mathrm{Conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$ and

$$\begin{aligned}&\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x}) :=\\ &\qquad \max\{t \in \mathbb{R} \mid (\boldsymbol{x}, t) \in \mathrm{Conv}\{(\boldsymbol{x}_1, \varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)), \ldots, (\boldsymbol{x}_r, \varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))\} \subseteq \mathbb{R}^n \times \mathbb{R}\}\end{aligned}$$

for $\boldsymbol{x} \in \Theta$ (see Conventions and terminology 2 for the definition of $\widetilde{\boldsymbol{x}}_1, \ldots, \widetilde{\boldsymbol{x}}_r$). Using the above points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$, we can construct a birational morphisms $\mu : Y \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth and normal arithmetic varieties, and a nef arithmetic $\mathbb{Q}$-divisor $\overline{P}$ of $(C^\infty \cap \mathrm{PSH})$-type on Y such that

$$\overline{P} \leq \mu^*(\overline{D}_{\boldsymbol{a}}) \quad \text{and} \quad \widehat{\mathrm{vol}}(\overline{P}) > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon.$$

For details, see Section 6.

I would like to express my thanks to Prof. Yuan. The studies of this paper started from his question. I thank Dr. Uchida. Without his calculation of the limit of a sequence, I could not find the positive part of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$. In addition, I also thank Dr. Hajli for his comments.

Conventions and terminology.

1. For $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$, the i-th entry x_i of $\boldsymbol{x}$ is denoted by $\boldsymbol{x}(i)$. We define $|\boldsymbol{x}|$ to be $|\boldsymbol{x}| := x_1 + \cdots + x_r$.

2. For $\boldsymbol{x} = (x_1, \ldots, x_r) \in \mathbb{R}^r$ and $m \in \mathbb{R}$, we define $\widetilde{\boldsymbol{x}}^m \in \mathbb{R}^{r+1}$ to be

$$\widetilde{\boldsymbol{x}}^m = (m - x_1 - \cdots - x_r, x_1, \ldots, x_r).$$

Note that $|\widetilde{\boldsymbol{x}}^m| = m$. For simplicity, in the case where $m = 1$, we denote $\widetilde{\boldsymbol{x}}^m$ by $\widetilde{\boldsymbol{x}}$.

3. Let $\boldsymbol{e} = (e_1, \ldots, e_r) \in \mathbb{Z}^r_{\geq 0}$ and $l = |\boldsymbol{e}|$. A monomial $z_1^{e_1} \cdots z_r^{e_r}$ is denoted by $z^{\boldsymbol{e}}$. The multinomial coefficient $\dfrac{l!}{e_1! \cdots e_r!}$ is denoted by $\dbinom{l}{\boldsymbol{e}}$.

4. We freely use the notations in the paper [9].

1. Fundamental properties of the characteristic function

Let $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, T_1, \dots, T_n])$, $H_i = \{T_i = 0\}$ and $z_i = T_i/T_0$ for $i = 0, \dots, n$. Let us fix $\boldsymbol{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}_{>0}$. We set

$$h_{\boldsymbol{a}} = a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2, \quad g_{\boldsymbol{a}} = \log h_{\boldsymbol{a}} \quad \text{and} \quad \omega_{\boldsymbol{a}} = dd^c(g_{\boldsymbol{a}})$$

on $\mathbb{P}^n(\mathbb{C})$, that is,

$$g_{\boldsymbol{a}} = -\log|T_0|^2 + \log\left(a_0|T_0|^2 + \cdots + a_n|T_n|^2\right).$$

Proposition 1.1. (1) *$\omega_{\boldsymbol{a}}$ is positive. In particular, $g_{\boldsymbol{a}}$ is a H_0-Green function of $(C^\infty \cap \mathrm{PSH})$-type.*

(2) *If we set $\Phi_{\boldsymbol{a}} = \omega_{\boldsymbol{a}}^{\wedge n}$, then*

$$\Phi_{\boldsymbol{a}} = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \frac{n! a_0 \cdots a_n}{h_{\boldsymbol{a}}^{n+1}} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$

and $\displaystyle\int_{\mathbb{P}^n(\mathbb{C})} \Phi_{\boldsymbol{a}} = 1.$

Proof. (1) Note that

$$\omega_{\boldsymbol{a}} = \frac{\sqrt{-1}}{2\pi}\left(\sum_{i=1}^n \frac{a_i}{h_{\boldsymbol{a}}(z)} dz_i \wedge d\bar{z}_i - \sum_{i,j} \frac{a_i a_j \bar{z}_i z_j}{h_{\boldsymbol{a}}(z)^2} dz_i \wedge d\bar{z}_j\right).$$

If we set

$$A = \left(\delta_{ij}\frac{a_i}{h_{\boldsymbol{a}}(z)} - \frac{a_i a_j \bar{z}_i z_j}{h_{\boldsymbol{a}}(z)^2}\right)_{\substack{1 \le i \le n, \\ 1 \le j \le n}},$$

then it is easy to see that

$$\begin{pmatrix} \bar{\lambda}_1 & \cdots & \bar{\lambda}_n \end{pmatrix} A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \frac{a_0 \sum_{i=1}^n a_i |\lambda_i|^2 + \sum_{i<j} a_i a_j |z_i \bar{\lambda}_j - z_j \bar{\lambda}_i|^2}{h_{\boldsymbol{a}}(z)^2}.$$

Thus $\omega_{\boldsymbol{a}}$ is positive definite.

(2) The first assertion follows from the following claim:

Claim 1.1.1. *For $\alpha_1, \dots, \alpha_n \in \mathbb{C}$,*

$$\det\left(\delta_{ij} t_i - \alpha_i \bar{\alpha}_j\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}} = t_1 \cdots t_n - \sum_{i=1}^n |\alpha_i|^2 t_1 \cdots t_{i-1} \cdot t_{i+1} \cdots t_n.$$

Proof. We denote $\left(\delta_{ij} t_i - \alpha_i \bar{\alpha}_j\right)_{\substack{1 \le i \le n \\ 1 \le j \le n}}$ by B. If $t_i = t_j = 0$ for $i \neq j$, then the i-the column and the j-the column of B are linearly dependent, so that $\det B = 0$. Therefore, we can set

$$\det B = t_1 \cdots t_n - \sum_{i=1}^n c_i t_1 \cdots t_{i-1} \cdot t_{i+1} \cdots t_n$$

for some $c_1, \dots, c_n \in \mathbb{C}$. It is easy to see that $\det B = -|\alpha_i|^2$ if $t_i = 0$ and $t_1 = \cdots = t_{i-1} = t_{i+1} = \cdots = t_n = 1$. Thus $c_i = |\alpha_i|^2$. $\square$

Let $|\cdot|_{\boldsymbol{a}}$ be a C^{∞}-hermitian metric of $\mathcal{O}(1)$ given by

$$|T_i|_{\boldsymbol{a}} = \frac{|T_i|}{\sqrt{a_0|T_1|^2 + a_1|T_1|^2 + \cdots + a_n|T_n|^2}}$$

for $i = 0, \ldots, n$. Then $c_1(\mathcal{O}(1), |\cdot|_{\boldsymbol{a}}) = \omega_{\boldsymbol{a}}$. Thus the second assertion follows. $\square$

We define a function $\varphi_{\boldsymbol{a}} : \mathbb{R}^{n+1}_{\geq 0} \to \mathbb{R}$ to be

$$\varphi_{\boldsymbol{a}}(x_0, \ldots, x_n) = -\sum_{i=0}^{n} x_i \log x_i + \sum_{i=0}^{n} x_i \log a_i,$$

which is called the *characteristic function of $g_{\boldsymbol{a}}$*. The function $\varphi_{\boldsymbol{a}}$ play a key role in this paper. Here note that $\varphi_{\boldsymbol{a}}(0, \ldots, \overset{\overset{i}{\vee}}{1}, \ldots, 0) = \log a_i$ for $i = 0, \ldots, n$. Notably the characteristic function is very similar to the entropy function in the coding theory.

Lemma 1.2. *For $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0}$ with $x_0 + x_1 + \cdots + x_n = 1$,*

$$\varphi_{\boldsymbol{a}}(x_0, \ldots, x_n) \leq \log(a_0 + a_1 + \cdots + a_n),$$

and the equality holds if and only if

$$x_0 = a_0/(a_0 + a_1 + \cdots + a_n), \ldots, x_n = a_n/(a_0 + a_1 + \cdots + a_n).$$

Proof. Let us begin with the following claim:

Claim 1.2.1. *For $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r, t_1, \ldots, t_r \in \mathbb{R}_{>0}$ with $\alpha_1 + \cdots + \alpha_r = 1$,*

$$\sum_{i=1}^{r} \alpha_i \log t_i \leq \log\left(\sum_{i=1}^{r} \beta_i t_i\right) + \sum_{i=1}^{r} \alpha_i \log \frac{\alpha_i}{\beta_i},$$

and the equality holds if and only if $\frac{\beta_1}{\alpha_1} t_1 = \cdots = \frac{\beta_r}{\alpha_r} t_r$.

Proof. Note that if we set $t_i' = \frac{\beta_i}{\alpha_i} t_i$ for $i = 1, \ldots, r$, then

$$\sum_{i=1}^{r} \alpha_i \log t_i - \log\left(\sum_{i=1}^{r} \beta_i t_i\right) = \sum_{i=1}^{r} \alpha_i \log t_i' - \log\left(\sum_{i=1}^{r} \alpha_i t_i'\right) + \sum_{i=1}^{r} \alpha_i \log \frac{\alpha_i}{\beta_i}.$$

Thus we may assume that $\alpha_i = \beta_i$ for all i. In this case, the inequality is nothing more than Jensen's inequality for the strictly concave function $\log$. $\square$

We set $I = \{i \mid x_i \neq 0\}$. Then, using the above claim, we have

$$\sum_{i \in I} x_i \log a_i \leq \log\left(\sum_{i \in I} a_i\right) + \sum_{i \in I} x_i \log x_i,$$

and hence

$$\begin{aligned} \varphi_{\boldsymbol{a}}(x_0, \ldots, x_n) &= \sum_{i \in I} -x_i \log x_i + \sum_{i \in I} x_i \log a_i \\ &\leq \log\left(\sum_{i \in I} a_i\right) \leq \log(a_0 + \cdots + a_n). \end{aligned}$$

In addition, the equality holds if and only if $a_i/x_i = a_j/x_j$ for all $i, j \in I$ and $a_i = 0$ for all $i \notin I$. Thus the assertion follows. $\square$

Note that

$$H^0(\mathbb{P}^n_{\mathbb{Z}}, lH_0) = \bigoplus_{\boldsymbol{e}\in\mathbb{Z}^n_{\geq 0}, |\boldsymbol{e}|\leq l} \mathbb{Z}z^{\boldsymbol{e}}$$

(for the definition of $|\boldsymbol{e}|$ and $z^{\boldsymbol{e}}$, see Conventions and terminology 1 and 3). According as [9], $|\cdot|_{lg_{\boldsymbol{a}}}$, $\|\cdot\|_{lg_{\boldsymbol{a}}}$ and $\langle\cdot,\cdot\rangle_{lg_{\boldsymbol{a}}}$ are defined by

$$|\phi|_{lg_{\boldsymbol{a}}} := |\phi|\exp(-lg_{\boldsymbol{a}}/2), \quad \|\phi\|_{lg_{\boldsymbol{a}}} := \sup\{|\phi|_{lg_{\boldsymbol{a}}}(x) \mid x \in \mathbb{P}^n(\mathbb{C})\}$$

and

$$\langle\phi,\psi\rangle_{lg_{\boldsymbol{a}}} := \int_{\mathbb{P}^n(\mathbb{C})} \phi\bar{\psi}\exp(-lg_{\boldsymbol{a}})\Phi_{\boldsymbol{a}},$$

where $\phi, \psi \in H^0(\mathbb{P}^n(\mathbb{C}), lH_0)$.

Proposition 1.3. *Let l be a positive integer and $\boldsymbol{e} = (e_1,\ldots,e_n), \boldsymbol{e}' = (e'_1,\ldots,e'_n) \in \mathbb{Z}^n_{\geq 0}$ with $|\boldsymbol{e}|, |\boldsymbol{e}'| \leq l$.*

(1) $\|z^{\boldsymbol{e}}\|^2_{lg_{\boldsymbol{a}}} = \exp(-l\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{e}}^l/l))$ *(for the definition of $\widetilde{\boldsymbol{e}}^l$, see Conventions and terminology* 2*).*

(2)
$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}'}\rangle_{lg_{\boldsymbol{a}}} = \begin{cases} 0 & \textit{if } \boldsymbol{e}\neq\boldsymbol{e}', \\ \dfrac{1}{\binom{n+l}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l}} & \textit{if } \boldsymbol{e}=\boldsymbol{e}' \end{cases}$$

(for the definition of $\binom{l}{\widetilde{\boldsymbol{e}}^l}$, see Conventions and terminology 3*).*

Proof. (1) By the definition of $|z^{\boldsymbol{e}}|_{lg_{\boldsymbol{a}}}$, we can see

$$\log|z^{\boldsymbol{e}}|^2_{lg_{\boldsymbol{a}}} = e_0\log|T_0|^2 + \cdots + e_n\log|T_n|^2 - l\log(a_0|T_0|^2 + \cdots + a_n|T_n|^2),$$

where $e_0 = l - e_1 - \cdots - e_n$ and $(T_0 : \cdots : T_n)$ is a homogeneous coordinate of $\mathbb{P}^n(\mathbb{C})$ such that $z_i = T_i/T_0$. Here we set $e'_i = e_i/l$ for $i = 0,\ldots,l$ and $I = \{i \mid e_i \neq 0\}$. Then, by using Claim 1.2.1,

$$\frac{1}{l}\log|z^{\boldsymbol{e}}|^2_{lg_{\boldsymbol{a}}} \leq \sum_{i\in I} e'_i\log|T_i|^2 - \log\left(\sum_{i\in I} a_i|T_i|^2\right) \leq -\varphi_{\boldsymbol{a}}(e'_0,\ldots,e'_n).$$

Moreover, if we set $T_i = \sqrt{e'_i/a_i}$ for $i = 0,\ldots,n$, then the equality holds. Thus (1) follows.

(2) First of all, Proposition 1.1,

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}'}\rangle_{lg_{\boldsymbol{a}}} = \left(\frac{\sqrt{-1}}{2\pi}\right)^n \int_{\mathbb{P}^n(\mathbb{C})} \frac{n!a_0\cdots a_n z^{\boldsymbol{e}}\bar{z}^{\boldsymbol{e}'} dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n}{(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2)^{n+l+1}}.$$

If we set $z_i = x_i^{1/2}\exp(2\pi\sqrt{-1}\theta_i)$, then the above integral is equal to

$$\int_{\mathbb{R}^n\times[0,1]^n} \frac{n!a_0\cdots a_n\prod_{i=1}^n x_i^{(e_i+e'_i)/2}\exp(2\pi\sqrt{-1}(e_i - e'_i))}{(a_0 + a_1x_1 + \cdots + a_nx_n)^{n+l+1}} dx_1\cdots dx_n d\theta_1\cdots d\theta_n,$$

and hence

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}'}\rangle_{lg_{\boldsymbol{a}}} = \begin{cases} 0 & \text{if } \boldsymbol{e} \neq \boldsymbol{e}', \\ \displaystyle\int_{\mathbb{R}^n} \frac{n! a_0 \cdots a_n x_1^{e_1} \cdots x_n^{e_n}}{(a_0 + a_1 x_1 + \cdots + a_n x_n)^{n+l+1}} dx_1 \cdots dx_n & \text{if } \boldsymbol{e} = \boldsymbol{e}'. \end{cases}$$

It is easy to see that

$$\int_0^\infty \frac{ax^m}{(ax+b)^n} dx = \frac{m!}{a^m b^{n-m-1}(n-1)(n-2)\cdots(n-m)(n-m-1)}$$

for $a, b \in \mathbb{R}_{>0}$ and $n, m \in \mathbb{Z}_{\geq 0}$ with $n - m \geq 2$. Thus we can see

$$\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}}\rangle_{lg_{\boldsymbol{a}}} = \frac{n! e_n! \cdots e_1!}{(n+l)(n+l-1)\cdots(e_0+1) a_n^{e_n} \cdots a_1^{e_1} a_0^{e_0}},$$

where $e_0 = l - e_1 - \cdots - e_n$. Therefore the assertion follows. □

Next we observe the following lemma:

Lemma 1.4. *If we set $A_n = (n+2)/2$ and $B_n = (n+2)\log\sqrt{2\pi} + (n+2)/12$, then*

$$\left| \frac{1}{l} \log\left(\frac{l!}{k_0! \cdots k_n!} a_0^{k_0} \cdots a_n^{k_n} \right) - \varphi_{\boldsymbol{a}}(k_0/l, \ldots, k_n/l) \right| \leq \frac{1}{l}(A_n \log l + B_n)$$

holds for all $l \geq 1$ and $(k_0, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ with $k_0 + \cdots + k_n = l$.

Proof. First of all, note that, for $m \geq 1$,

$$m! = \sqrt{2\pi m}\, \frac{m^m}{e^m}\, e^{\frac{\theta_m}{12m}} \quad (0 < \theta_m < 1)$$

by Stirling's formula. We set $I = \{i \mid k_i \neq 0\}$. Then

$$\log(l!) = \log(\sqrt{2\pi l}) + l \log l - l + \frac{\theta_l}{12l},$$
$$\log(k_i!) = \log(\sqrt{2\pi k_i}) + k_i \log k_i - k_i + \frac{\theta_{k_i}}{12k_i} \quad (i \in I).$$

Therefore,

$$\frac{1}{l}\log\left(\frac{l!}{k_0!\cdots k_n!} a_0^{k_0}\cdots a_n^{k_n}\right) = \varphi_{\boldsymbol{a}}(k_0/l, \ldots, k_n/l) + \frac{1}{l}\log(\sqrt{2\pi l}) + \frac{\theta_l}{12l^2} - \sum_{i \in I}\left(\frac{1}{l}\log(\sqrt{2\pi k_i}) + \frac{\theta_{k_i}}{12lk_i}\right),$$

which yields the assertion. □

Let $\overline{D}_{\boldsymbol{a}}$ be an arithmetic divisor of $(C^\infty \cap \mathrm{PSH})$-type on $\mathbb{P}^n_{\mathbb{Z}}$ given by

$$\overline{D}_{\boldsymbol{a}} := (H_0, g_{\boldsymbol{a}}) = (H_0, \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2)).$$

Moreover, for $\lambda \in \mathbb{R}$, $\Theta_{\boldsymbol{a},\lambda}$ is defined to be

$$\Theta_{\boldsymbol{a},\lambda} := \{(x_1, \ldots, x_n) \in \Delta_n \mid \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \geq \lambda\},$$

where $\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\}$. Note that $\Theta_{\boldsymbol{a},\lambda}$ is a compact convex set. For simplicity, we denote $\Theta_{\boldsymbol{a},0}$ by $\Theta_{\boldsymbol{a}}$, that is,

$$\Theta_{\boldsymbol{a}} = \{(x_1, \ldots, x_n) \in \Delta_n \mid \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \geq 0\},$$

Finally we consider the following proposition:

Proposition 1.5. *Let us fix a positive integer l. Then we have the following:*

(1) *$l\Theta_{\boldsymbol{a},\lambda} \cap \mathbb{Z}^n \neq \emptyset$ if and only if there is a non-zero rational function ϕ on $\mathbb{P}^n_{\mathbb{Z}}$ such that $lH_0 + (\phi) \geq 0$ and $\|\phi\|_{lg_{\boldsymbol{a}}} \leq e^{-l\lambda}$.*

(2) *If $l\Theta_{\boldsymbol{a},\lambda} \cap \mathbb{Z} \neq \emptyset$, then*

$$\left\langle \{\phi \in \mathrm{Rat}(\mathbb{P}^n_{\mathbb{Z}})^{\times} \mid lH_0 + (\phi) \geq 0,\ \|\phi\|_{lg_{\boldsymbol{a}}} \leq e^{-l\lambda}\} \right\rangle_{\mathbb{Z}} = \bigoplus_{\boldsymbol{e} \in l\Theta_{\boldsymbol{a},\lambda} \cap \mathbb{Z}^n} \mathbb{Z} z^{\boldsymbol{e}}.$$

Proof. Let us begin with the following claim:

Claim 1.5.1. *Let ϕ be a non-zero rational function on $\mathbb{P}^n_{\mathbb{Z}}$ such that $lH_0 + (\phi) \geq 0$ and $\|\phi\|_{lg_{\boldsymbol{a}}} \leq e^{-l\lambda}$. If we write*

$$\phi = \sum_{\boldsymbol{e} \in \mathbb{Z}^n_{\geq 0}, |\boldsymbol{e}| \leq l} c_{\boldsymbol{e}} z^{\boldsymbol{e}} \quad (c_{\boldsymbol{e}} \in \mathbb{Z}),$$

then $\{\boldsymbol{e} \mid c_{\boldsymbol{e}} \neq 0\} \subseteq l\Theta_{\boldsymbol{a},\lambda}$.

Proof. Clearly we may assume that $\phi \neq 0$. We set $\{\boldsymbol{e} \mid c_{\boldsymbol{e}} \neq 0\} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$, where $\boldsymbol{e}_i \neq \boldsymbol{e}_j$ for $i \neq j$. Let $\boldsymbol{e}_i$ be an extreme point of $\mathrm{Conv}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$. Here let us see that $\boldsymbol{e}_i \in l\Theta_{\boldsymbol{a},\lambda}$. Renumbering $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$, we may assume that $i = 1$. Then, for $k \geq 1$,

$$\phi^k = c_{\boldsymbol{e}_1}^k z^{k\boldsymbol{e}_1} + \sum_{\substack{k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}, \\ k_1 + \cdots + k_m = k,\ k_1 \neq k}} \frac{k!}{k_1! \cdots k_m!} c_{\boldsymbol{e}_1}^{k_1} \cdots c_{\boldsymbol{e}_m}^{k_m} z^{k_1\boldsymbol{e}_1 + \cdots + k_m\boldsymbol{e}_m}.$$

Let us check that $k\boldsymbol{e}_1 \neq k_1\boldsymbol{e}_1 + \cdots + k_m\boldsymbol{e}_m$ holds for all $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ with $k_1 + \cdots + k_m = k$ and $k_1 \neq k$. Otherwise, $\boldsymbol{e}_1 = (k_2/(k-k_1))\boldsymbol{e}_2 + \cdots + (k_m/(k-k_1))\boldsymbol{e}_m$. This is a contradiction because $\boldsymbol{e}_1$ is an extreme point of $\mathrm{Conv}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$. Therefore, we can write

$$\phi^k = c_{\boldsymbol{e}_1}^k z^{k\boldsymbol{e}_1} + \sum_{\boldsymbol{e}' \in \mathbb{Z}^n_{\geq 0}, \boldsymbol{e}' \neq k\boldsymbol{e}_1} c'_{\boldsymbol{e}'} z^{\boldsymbol{e}'}$$

for some $c'_{\boldsymbol{e}'} \in \mathbb{Z}$, which implies

$$\langle \phi^k, \phi^k \rangle_{klg_{\boldsymbol{a}}} = \frac{c_{\boldsymbol{e}_1}^{2k}}{\binom{kl+n}{n}\binom{kl}{k\widetilde{\boldsymbol{e}}_1^l}\boldsymbol{a}^{k\widetilde{\boldsymbol{e}}_1^l}} + (\text{non-negative real number})$$

by Proposition 1.3. Since $\|\phi^k\|_{klg_a} \leq e^{-\lambda kl}$, we have $\langle \phi^k, \phi^k \rangle_{klg_{\boldsymbol{a}}} \leq e^{-\lambda kl}$, which yields

$$\binom{kl+n}{n}\binom{kl}{k\widetilde{\boldsymbol{e}}_1^l}\boldsymbol{a}^{k\widetilde{\boldsymbol{e}}_1^l} \geq e^{\lambda kl}.$$

Thus, by Lemma 1.4,

$$\varphi_{\boldsymbol{a}}\left(\frac{k\widetilde{\boldsymbol{e}}_1^l}{kl}\right) \geq \lambda - \frac{1}{kl}(A_n \log(kl) + B_n) - \frac{1}{kl}\log\binom{kl+n}{n}.$$

Therefore, by taking $k \to \infty$, $\varphi_{\boldsymbol{a}}\left(\frac{\widetilde{\boldsymbol{e}}_1^l}{l}\right) \geq \lambda$, and hence $\boldsymbol{e}_1 \in l\Theta_{\boldsymbol{a},\lambda}$.

Finally let us see the claim. Let $\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}$ be all extreme points of $\mathrm{Conv}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$. Then, by the above observation,

$$\mathrm{Conv}\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\} = \mathrm{Conv}\{\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}\} \subseteq l\Theta_{\boldsymbol{a},\lambda}$$

because $l\Theta_{\boldsymbol{a},\lambda}$ is a convex set. $\square$

Let us go back to the proofs of (1) and (2). By Proposition 1.3,

$$\|z^{\boldsymbol{e}}\|_{lg_{\boldsymbol{a}}} = \exp(-l\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{e}}^{l}/l)).$$

Thus (1) and (2) follow from the above claim. □

Remark 1.6. Let $\tilde{\rho}_{\boldsymbol{a}}$ be a hermitian inner product of $H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}_{\mathbb{P}^n}(1))$ given by

$$(\tilde{\rho}_{\boldsymbol{a}}(T_i, T_j))_{0 \leq i,j \leq n} = \begin{pmatrix} 1/a_0 & 0 & \cdots & 0 & 0 \\ 0 & 1/a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1/a_n \end{pmatrix}.$$

Let $\rho_{\boldsymbol{a}}$ be the quotient C^∞-hermitian metric of $\mathcal{O}_{\mathbb{P}^n}(1)$ induced by $\tilde{\rho}_{\boldsymbol{a}}$ and the canonical surjective homomorphism

$$H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1).$$

Then $g_{\boldsymbol{a}} = -\log \rho_{\boldsymbol{a}}(T_0, T_0)$.

Remark 1.7. Hajli [6] pointed out that, for $(x_1, \ldots, x_n) \in \Delta_n$,

$$-\varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n)$$

is the Legendre-Fenchel transform of $\log(a_0 + a_1 e^{u_1} + \cdots + a_n e^{u_n})$, that is,

$$\begin{aligned} &-\varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) \\ &\quad = \sup \left\{ u_1 x_1 + \cdots + u_n x_n - \log(a_0 + a_1 e^{u_1} + \cdots + a_n e^{u_n}) \mid (u_1, \ldots, u_n) \in \mathbb{R}^n \right\}. \end{aligned}$$

This can be easily checked by Claim 1.2.1.

2. Integral formula and Geography of $\overline{D}_{\boldsymbol{a}}$

Let X be a d-dimensional, generically smooth, normal and projective arithmetic variety. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X. Let Φ be an F_∞-invariant volume form on $X(\mathbb{C})$ with $\int_{X(\mathbb{C})} \Phi = 1$. Recall that $\langle \phi, \psi \rangle_g$ and $\|\phi\|_{g,L^2}$ are given by

$$\langle \phi, \psi \rangle_g := \int_{X(\mathbb{C})} \phi \bar{\psi} \exp(-g) \Phi \quad \text{and} \quad \|\phi\|_{g,L^2} := \sqrt{\langle \phi, \phi \rangle_g}$$

for $\phi, \psi \in H^0(X, D)$. We set

$$\hat{H}^0_{L^2}(X, \overline{D}) := \{ \phi \in H^0(X, D) \mid \|\phi\|_{g,L^2} \leq 1 \}.$$

Let us begin with the following lemmas:

Lemma 2.1. $\widehat{\mathrm{vol}}(\overline{D}) = \lim_{l \to \infty} \dfrac{\log \# \hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!}$.

Proof. First of all, note that

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{l \to \infty} \frac{\log \# \hat{H}^0(X, l\overline{D})}{l^d/d!}$$

(cf. [9, Theorem 5.2.2]). Since $\hat{H}^0(X, l\overline{D}) \subseteq \hat{H}^0_{L^2}(X, l\overline{D})$, we have

$$\widehat{\mathrm{vol}}(\overline{D}) \leq \liminf_{l \to \infty} \frac{\log \# \hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!}.$$

On the other hand, by using Gromov's inequality (cf. [9, Proposition 3.1.1]), there is a constant C such that $\|\cdot\|_{\sup} \leq Cl^{d-1}\|\cdot\|_{L^2}$ on $H^0(X, lD)$. Thus, for any positive number ϵ, $\|\cdot\|_{\sup} \leq \exp(l\epsilon/2)\|\cdot\|_{L^2}$ holds for $l \gg 1$. This implies that

$$\hat{H}^0_{L^2}(X, l\overline{D}) \subseteq \hat{H}^0(X, l(\overline{D} + (0, \epsilon)))$$

for $l \gg 1$, which yields

$$\limsup_{l\to\infty} \frac{\log \#\hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!} \leq \widehat{\mathrm{vol}}(\overline{D} + (0, \epsilon)).$$

Therefore, by virtue of the continuity of $\widehat{\mathrm{vol}}$, we have

$$\limsup_{l\to\infty} \frac{\log \#\hat{H}^0_{L^2}(X, l\overline{D})}{l^d/d!} \leq \widehat{\mathrm{vol}}(\overline{D}),$$

and hence the lemma follows. □

Lemma 2.2. *Let Θ be a compact convex set in $\mathbb{R}^n$ such that* $\mathrm{vol}(\Theta) > 0$. *For each $l \in \mathbb{Z}_{\geq 1}$, let $A_l = (a_{\boldsymbol{e},\boldsymbol{e}'})_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n}$ be a positive definite symmetric real matrix indexed by $l\Theta \cap \mathbb{Z}^n$, and let K_l be a subset of $\mathbb{R}^{l\Theta\cap\mathbb{Z}^n} \simeq \mathbb{R}^{\#(l\Theta\cap\mathbb{Z}^n)}$ given by*

$$K_l = \left\{ (x_{\boldsymbol{e}}) \in \mathbb{R}^{l\Theta\cap\mathbb{Z}^n} \;\middle|\; \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \leq 1 \right\}.$$

We assume that there are positive constants C and D and a continuous function $\varphi : \Theta \to \mathbb{R}$ such that

$$\left| \log\left(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\right) - l\varphi\left(\frac{\boldsymbol{e}}{l}\right) \right| \leq C\log(l) + D$$

for all $l \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{e} \in l\Theta \cap \mathbb{Z}^n$. Then we have

$$\liminf_{l\to\infty} \frac{\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n})}{l^{n+1}} \geq \frac{1}{2}\int_\Theta \varphi(\boldsymbol{x}) d\boldsymbol{x}.$$

Moreover, if A_l is diagonal and all entries of A_l are less than or equal to 1 *(i.e., $a_{\boldsymbol{e},\boldsymbol{e}'} \leq 1$ $\forall \boldsymbol{e}, \boldsymbol{e}' \in l\Theta \cap \mathbb{Z}^n$) for each l, then*

$$\lim_{l\to\infty} \frac{\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n})}{l^{n+1}} = \frac{1}{2}\int_\Theta \varphi(\boldsymbol{x}) d\boldsymbol{x}.$$

Proof. By Minkowski's theorem,

$$\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n}) \geq \log(\mathrm{vol}(K_l)) - m_l \log(2),$$

where $m_l = \#(l\Theta \cap \mathbb{Z}^n)$. Note that

$$\log(\mathrm{vol}(K_l)) = -\frac{1}{2}\log(\det(A_l)) + \log V_{m_l},$$

where $V_r = \mathrm{vol}(\{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid x_1^2 + \cdots + x_r^2 \leq 1\})$. Moreover, by Hadamard's inequality,

$$\det(A_l) \leq \prod_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}}.$$

Thus

$$\log \#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n}) \geq \frac{1}{2}\sum_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \log\left(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\right) + \log V_{m_l} - m_l \log(2).$$

Further, there is a positive constant c_1 such that $m_l \leq c_1 l^n$ for $l \geq 1$. Thus we can see

$$\lim_{l\to\infty} \log(V_{m_l})/l^{n+1} = 0.$$

Therefore, it is sufficient to show that

$$\lim_{l\to\infty} \frac{1}{l^{n+1}} \sum_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \log\left(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\right) = \int_\Theta \varphi(\boldsymbol{x})d\boldsymbol{x}.$$

By our assumption, we have

$$\varphi\left(\frac{\boldsymbol{e}}{l}\right) - \frac{1}{l}(C\log l + D) \leq \frac{1}{l}\log\left(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\right) \leq \varphi\left(\frac{\boldsymbol{e}}{l}\right) + \frac{1}{l}(C\log l + D).$$

Note that

$$\lim_{l\to\infty} \frac{1}{l^n} \sum_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \varphi\left(\frac{\boldsymbol{e}}{l}\right) = \lim_{l\to\infty} \sum_{\boldsymbol{x}\in\Theta\cap(1/l)\mathbb{Z}^n} \varphi(\boldsymbol{x})\frac{1}{l^n} = \int_\Theta \varphi(\boldsymbol{x})d\boldsymbol{x}.$$

On the other hand, since $m_l \leq c_1 l^n$, we can see

$$\lim_{l\to\infty} \sum_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \frac{1}{l^{n+1}}(C\log l + D) = 0.$$

Thus the first assertion follows.

Next we assume that A_l is diagonal for each l. Then, since

$$K_l \subseteq \prod_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \left[-\sqrt{\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}}, \sqrt{\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}}\right],$$

we have

$$\log\#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n}) \leq \sum_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \log\left(2\sqrt{\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}} + 1\right).$$

Thus

$$\log\#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n}) \leq \frac{1}{2}\sum_{\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n} \log\left(\frac{1}{a_{\boldsymbol{e},\boldsymbol{e}}}\right) + m_l\log(3)$$

because $a_{\boldsymbol{e},\boldsymbol{e}} \leq 1$ and $2t+1 \leq 3t$ for $t \geq 1$. Therefore, as before,

$$\limsup_{l\to\infty} \frac{\log\#(K_l \cap \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n})}{l^{n+1}} \leq \frac{1}{2}\int_\Theta \varphi(\boldsymbol{x})d\boldsymbol{x}.$$

$\square$

From now on, we use the same notation as in Section 1. The purpose of this section is to prove the following theorem:

Theorem 2.3. (1) (*Integral formula*) *The following formulae hold:*

$$\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = \frac{(n+1)!}{2}\int_{\Theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n)dx_1\cdots dx_n,$$

and

$$\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \frac{(n+1)!}{2}\int_{\Delta_n} \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n)dx_1\cdots dx_n.$$

(2) $\overline{D}_{\boldsymbol{a}}$ *is ample if and only if* $\boldsymbol{a}(i) > 1$ *for all* $i = 0, \ldots, n$.
(3) $\overline{D}_{\boldsymbol{a}}$ *is nef if and only if* $\boldsymbol{a}(i) \geq 1$ *for all* $i = 0, \ldots, n$.

(4) $\overline{D}_{\boldsymbol{a}}$ *is big if and only if* $|\boldsymbol{a}| > 1$.
(5) $\overline{D}_{\boldsymbol{a}}$ *is pseudo-effective if and only if* $|\boldsymbol{a}| \geq 1$.
(6) *If* $|\boldsymbol{a}| = 1$, *then*

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) = \begin{cases} \{0, \pm z_1^{l\boldsymbol{a}(1)} \cdots z_n^{l\boldsymbol{a}(n)}\} & \text{if } l\boldsymbol{a} \in \mathbb{Z}^{n+1}, \\ \{0\} & \text{if } l\boldsymbol{a} \notin \mathbb{Z}^{n+1}. \end{cases}$$

(7) $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}})$ *if and only if* $\overline{D}_{\boldsymbol{a}}$ *is nef.*

Proof. First let us see the essential case of (1):

Claim 2.3.1. *If* $|\boldsymbol{a}| > 1$, *then* $\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = \dfrac{(n+1)!}{2} \displaystyle\int_{\Theta_{\boldsymbol{a}}} \varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{t}}) d\boldsymbol{t}$.

Proof. In this case, $\mathrm{vol}(\Theta_{\boldsymbol{a}}) > 0$. By using Proposition 1.5,

$$\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}) \subseteq \left\{ \phi \in \bigoplus_{\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \mathbb{Z} z^{\boldsymbol{e}} \ \middle|\ \langle \phi, \phi \rangle_{lg_{\boldsymbol{a}}} \leq 1 \right\} \subseteq \hat{H}^0_{L^2}(\mathbb{P}^n_{\mathbb{Z}}, l\overline{D}_{\boldsymbol{a}}),$$

which yields

$$\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = (n+1)! \lim_{l \to \infty} \frac{\log \# \left\{ \phi \in \bigoplus_{\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \mathbb{Z} z^{\boldsymbol{e}} \ \middle|\ \langle \phi, \phi \rangle_{lg_{\boldsymbol{e}}} \leq 1 \right\}}{l^{n+1}}$$

by Lemma 2.1. If we set

$$K_l = \left\{ (x_{\boldsymbol{e}}) \in \mathbb{R}^{l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \ \middle|\ \sum_{\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \frac{x_{\boldsymbol{e}}^2}{\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l} \boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l}} \leq 1 \right\},$$

then, by Proposition 1.3,

$$\# \left\{ \phi \in \bigoplus_{\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n} \mathbb{Z} z^{\boldsymbol{e}} \ \middle|\ \langle \phi, \phi \rangle_{lg_{\boldsymbol{a}}} \leq 1 \right\} = \#(K_l \cap \mathbb{Z}^{l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n}).$$

On the other hand, for $\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n$,

$$\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l} \boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l} = \frac{1}{\langle z^{\boldsymbol{e}}, z^{\boldsymbol{e}} \rangle_{lg_{\boldsymbol{a}}}} \geq \exp(l\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{e}}^l / l)) \geq 1.$$

Moreover, by Lemma 1.4, there are positive constants A and B such that

$$\left| \log \left(\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l} \boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l} \right) - l\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{e}}^l / l) \right| \leq A \log l + B$$

holds for all $l \in \mathbb{Z}_{\geq 1}$ and $\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n$. Thus the assertion follows from Lemma 2.2. $\square$

Next let us see the following claim:

Claim 2.3.2. *If* $s, t \in \mathbb{R}_{>0}$ *and* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha + \beta \neq 0$, *then*

$$\alpha \overline{D}_{t\boldsymbol{a}} + \beta \overline{D}_{s\boldsymbol{a}} = (\alpha + \beta) \overline{D}_{(t^\alpha s^\beta)^{\frac{1}{\alpha+\beta}} \boldsymbol{a}}.$$

Proof. This is a straightforward calculation. $\square$

(2) and (3): First of all, $\omega_{\boldsymbol{a}}$ is positive by Proposition 1.1. Let γ_i be a 1-dimensional closed subscheme given by $H_0 \cap \cdots \cap H_{i-1} \cap H_{i+1} \cap \cdots \cap H_n$. Then it is easy to see that $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}\big|_{\gamma_i}) = (1/2)\log(\boldsymbol{a}(i))$. Therefore we have "only if" part of (1) and (2).

We assume that $\boldsymbol{a}(i) > 1$ for all i. Then $\varphi_{\boldsymbol{a}}$ is positive on

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \mid x_0 + \cdots + x_n = 1\}.$$

Thus, for $\boldsymbol{e} \in \mathbb{Z}^n_{\geq 0}$ with $|\boldsymbol{e}| \leq 1$, $z^{\boldsymbol{e}}$ is a strictly small section by Proposition 1.3, which shows that $\overline{D}_{\boldsymbol{a}}$ is ample.

Next we assume that $\boldsymbol{a}(i) \geq 1$ for all i. Let γ be a 1-dimensional closed integral subscheme of $\mathbb{P}^n_{\mathbb{Z}}$. Then we can find H_i such that $\gamma \not\subseteq H_i$. Note that

$$\overline{D}_{\boldsymbol{a}} + \widehat{(z_i)} = (H_i, \log(\boldsymbol{a}(0)|w_0|^2 + \cdots + \boldsymbol{a}(n)|w_n|^2)),$$

where $w_k = T_k/T_i$ ($k = 0, \ldots, n$). Therefore $\widehat{\deg}(\overline{D}_{\boldsymbol{a}}\big|_{\gamma}) \geq 0$ because

$$\log(\boldsymbol{a}(0)|w_0|^2 + \cdots + \boldsymbol{a}(n)|w_n|^2) \geq 0.$$

(6): In this case, $\Theta_{\boldsymbol{a}} = \{(\boldsymbol{a}(1), \ldots, \boldsymbol{a}(n))\}$ and $\varphi_{\boldsymbol{a}}(\boldsymbol{a}) = 0$ by Lemma 1.2. Moreover, if $l\boldsymbol{a} \in \mathbb{Z}^{n+1}$, then

$$\|z^{l(\boldsymbol{a}(1),\ldots,\boldsymbol{a}(n))}\|^2_{lg_{\boldsymbol{a}}} = \exp(-l\varphi_{\boldsymbol{a}}(\boldsymbol{a})) = 1$$

by Proposition 1.3. Thus the assertion follows from Proposition 1.5.

(4) and (5): By using (6), in order to see (4) and (5), it is sufficient to show the following:

(i) $\overline{D}_{\boldsymbol{a}}$ is big if $|\boldsymbol{a}| > 1$.
(ii) $\overline{D}_{\boldsymbol{a}}$ is pseudo-effective if $|\boldsymbol{a}| \geq 1$.
(iii) $\overline{D}_{\boldsymbol{a}}$ is not pseudo-effective if $|\boldsymbol{a}| < 1$.

(i) It follows from Claim 2.3.1 because $\operatorname{vol}(\Theta_{\boldsymbol{a}}) > 0$.

(ii) We choose a real number t such that $t > 1$ and $\overline{D}_{t\boldsymbol{a}}$ is ample. By Claim 2.3.2,

$$\overline{D}_{\boldsymbol{a}} + \epsilon\overline{D}_{t\boldsymbol{a}} = (1+\epsilon)\overline{D}_{t^{\frac{\epsilon}{1+\epsilon}}\boldsymbol{a}}.$$

For any $\epsilon > 0$, since $t^{\frac{\epsilon}{1+\epsilon}}|\boldsymbol{a}| > 1$, $(1+\epsilon)\overline{D}_{t^{\frac{\epsilon}{1+\epsilon}}\boldsymbol{a}}$ is big by (i), which shows that $\overline{D}_{\boldsymbol{a}}$ is pseudo-effective.

(iii) Let us choose a positive real number t such that $\overline{D}_{t\boldsymbol{a}}$ is ample. We also choose a positive number ϵ such that if we set $\boldsymbol{a}' = t^{\frac{\epsilon}{1+\epsilon}}\boldsymbol{a}$, then $|\boldsymbol{a}'| < 1$. We assume that $\overline{D}_{\boldsymbol{a}}$ is pseudo-effective. Then

$$\overline{D}_{\boldsymbol{a}} + \epsilon\overline{D}_{t\boldsymbol{a}} = (1+\epsilon)\overline{D}_{\boldsymbol{a}'}$$

is big by [9, Proposition 6.3.2], which means that $\overline{D}_{\boldsymbol{a}'}$ is big. On the other hand, as $|\boldsymbol{a}'| < 1$, we have $\Theta_{\boldsymbol{a}'} = \emptyset$. Thus $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, n\overline{D}_{\boldsymbol{a}'}) = \{0\}$ for all $n \geq 1$ by Proposition 1.5. This is a contradiction.

(1): For the first formula, we may assume that $|\boldsymbol{a}| \leq 1$ by Claim 2.3.1. In this case, $\overline{D}_{\boldsymbol{a}}$ is not big by (4) and $\Theta_{\boldsymbol{a}}$ is either $\emptyset$ or $\{(a_1, \ldots, a_n)\}$. Thus the assertion follows. For the second formula, the arithmetic Hilbert-Samuel formula (cf. [4] and [1]) yields

$$\frac{\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1})}{(n+1)!} = \lim_{l\to\infty} \frac{\widehat{\chi}\left(H^0(\mathbb{P}^n_{\mathbb{Z}}, lH_0), \langle\ ,\ \rangle_{lg_a}\right)}{l^{n+1}}.$$

On the other hand,

$$\widehat{\chi}\left(H^0(\mathbb{P}^n_{\mathbb{Z}}, lH_0), \langle\ ,\ \rangle_{lg_a}\right) = \sum_{\boldsymbol{e}\in l\Delta_n\cap\mathbb{Z}^n} \log\left(\sqrt{\binom{l+n}{n}\binom{l}{\widetilde{\boldsymbol{e}}^l}\boldsymbol{a}^{\widetilde{\boldsymbol{e}}^l}}\right) + \log V_{\#(l\Delta_n\cap\mathbb{Z}^n)}.$$

Thus, in the same way as the proof of Lemma 2.2 and Claim 2.3.1, we can see the second formula.

(7): It follows from (1) and (3). ☐

Finally let us consider the following proposition:

Proposition 2.4. *For any positive integer l, there exists $\boldsymbol{a} \in \mathbb{Q}^{n+1}_{>0}$ such that $|\boldsymbol{a}| > 1$ and that $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, k\overline{D}_{\boldsymbol{a}}) = \{0\}$ for $k = 1, \ldots, l$.*

Proof. Let us choose positive rational numbers $a'_1, \ldots, a'_n$ such that $a'_1 + \cdots + a'_n < 1$ and $a'_1 < 1/l, \ldots, a'_n < 1/l$. We set $a'_0 = 1 - a'_1 - \cdots - a'_n$ and $\boldsymbol{a}' = (a'_0, \ldots, a'_n)$. Moreover, for a rational number $\lambda > 1$, we set

$$K_\lambda = \{\boldsymbol{x} \in \Delta_n \mid \varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}) + \log\lambda \geq 0\},$$

where $\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\}$.

Claim 2.4.1. *We can find a rational number $\lambda > 1$ such that $K_\lambda \subseteq (0, 1/l)^n$.*

Proof. We assume that $K_{1+(1/m)} \not\subseteq (0, 1/l)^n$ for all $m \in \mathbb{Z}_{\geq 1}$, that is, we can find $\boldsymbol{x}_m \in K_{1+(1/m)} \setminus (0, 1/l)^n$ for each $m \geq 1$. Since Δ_n is compact, there is a subsequence $\{\boldsymbol{x}_{m_i}\}$ of $\{\boldsymbol{x}_m\}$ such that $\boldsymbol{x} = \lim_{i\to\infty} \boldsymbol{x}_{m_i}$ exists. Note that $\boldsymbol{x} \notin (0, 1/l)^n$ because $\boldsymbol{x}_{m_i} \notin (0, 1/l)^n$ for all i. On the other hand, since $\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_{m_i}) + \log(1 + (1/m_i)) \geq 0$ for all i, we have $\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}) \geq 0$, and hence $\boldsymbol{x} = (a'_1, \ldots, a'_n)$ by Lemma 1.2. This is a contradiction. ☐

We choose a rational number $\lambda > 1$ as in the above claim. Here we set $\boldsymbol{a} = \lambda\boldsymbol{a}'$. Then, as $\varphi_{\boldsymbol{a}} = \varphi_{\boldsymbol{a}'} + \log\lambda$, we have $\Theta_{\boldsymbol{a}} \subseteq (0, 1/l)^n$. We assume that $\hat{H}^0(\mathbb{P}^n_{\mathbb{Z}}, k\overline{D}_{\boldsymbol{a}}) \neq \{0\}$ for some k with $1 \leq k \leq l$. Then, by Proposition 1.5, there is $\boldsymbol{e} = (e_1, \ldots, e_n) \in k\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n$, that is, $\boldsymbol{e}/k \in \Theta_{\boldsymbol{a}}$. Thus $0 < e_i/k < 1/l$ for all i. This is a contradiction. ☐

3. Asymptotic multiplicity

Let X be a d-dimensional, projective, generically smooth and normal arithmetic variety. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X. We set

$$N(\overline{D}) = \left\{ l \in \mathbb{Z}_{>0} \mid \hat{H}^0(X, l\overline{D}) \neq \{0\} \right\}.$$

We assume that $N(\overline{D}) \neq \emptyset$. Then $\mu_x(\overline{D})$ for $x \in X$ is defined to be

$$\mu_x(\overline{D}) := \inf\left\{ \mathrm{mult}_x(D + (1/l)(\phi)) \mid l \in N(\overline{D}),\ \phi \in \hat{H}^0(X, l\overline{D}) \setminus \{0\} \right\},$$

which is called the *asymptotic multiplicity of $\overline{D}$ at x*. The following proposition is the fundamental properties of the asymptotic multiplicity.

Proposition 3.1 ([9, Proposition 6.5.2 and Proposition 6.5.3])**.** *Let $\overline{D}$ and $\overline{E}$ be arithmetic $\mathbb{R}$-divisors of C^0-type such that $N(\overline{D}) \neq \emptyset$ and $N(\overline{E}) \neq \emptyset$. Then we have the following:*

(1) $\mu_x(\overline{D} + \overline{E}) \leq \mu_x(\overline{D}) + \mu_x(\overline{E})$.
(2) *If $\overline{D} \leq \overline{E}$, then* $\mu_x(\overline{E}) \leq \mu_x(\overline{D}) + \mathrm{mult}_x(E - D)$.
(3) $\mu_x(\overline{D} + \widehat{(\phi)}) = \mu_x(\overline{D})$ *for* $\phi \in \mathrm{Rat}(X)^\times$.
(4) $\mu_x(a\overline{D}) = a\mu_x(\overline{D})$ *for* $a \in \mathbb{Q}_{>0}$.

(5) *If $\overline{D}$ is nef and big, then $\mu_x(\overline{D}) = 0$.*

Moreover, we have the following lemma.

Lemma 3.2. *For each $l \in N(\overline{D})$, let $\{\phi_{l,1}, \ldots, \phi_{l,r_l}\}$ be a subset of $\hat{H}^0(X, l\overline{D}) \setminus \{0\}$ such that $\hat{H}^0(X, l\overline{D}) \subseteq \langle \phi_{l,1}, \ldots, \phi_{l,r_l} \rangle_{\mathbb{Z}}$. Let x be a point of X such that the Zariski closure $\overline{\{x\}}$ of $\{x\}$ is flat over $\mathbb{Z}$. Then*

$$\mu_x(\overline{D}) = \inf\{\operatorname{mult}_x(D + (1/l)(\phi_{l,i})) \mid l \in N(D),\ i = 1, \ldots, r_l\}.$$

Proof. Clearly

$$\mu_x(\overline{D}) \leq \inf\{\operatorname{mult}_x(D + (1/l)(\phi_{l,i})) \mid l \in N(D),\ i = 1, \ldots, r_l\}.$$

Let us consider the converse inequality. For $l \in N(\overline{D})$ and $\phi \in \hat{H}^0(X, l\overline{D}) \setminus \{0\}$, we set $\phi = \sum_{i=1}^{r_l} c_i \phi_{l,i}$ for some $c_1, \ldots, c_{r_l} \in \mathbb{Z}$. Note that

$$\operatorname{mult}_x((\phi + \psi)) \geq \min\{\operatorname{mult}_x((\phi)), \operatorname{mult}_x((\psi))\} \quad \text{and} \quad \operatorname{mult}_x((a)) = 0$$

for $\phi, \psi \in \operatorname{Rat}(X)^{\times}$ and $a \in \mathbb{Q}^{\times}$ with $\phi + \psi \neq 0$. Thus we can find i such that

$$\operatorname{mult}_x((\phi)) \geq \operatorname{mult}_x((\phi_{l,i})),$$

and hence the converse inequality holds. □

4. Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$

We use the same notation as in Section 1. We assume $n = 1$. In this section, we consider the Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1])$. Note that $\Theta_{\boldsymbol{a}}$ is a closed interval in $[0, 1]$. For simplicity, we denote the affine coordinate z_1 by z, that is, $z = T_1/T_0$.

Theorem 4.1. *The Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ exists if and only if $a_0 + a_1 \geq 1$. Moreover, if we set $\vartheta_{\boldsymbol{a}} = \inf \Theta_{\boldsymbol{a}}$, $\theta_{\boldsymbol{a}} = \sup \Theta_{\boldsymbol{a}}$, $P_{\boldsymbol{a}} = \theta_{\boldsymbol{a}} H_0 - \vartheta_{\boldsymbol{a}} H_1$ and*

$$p_{\boldsymbol{a}}(z) = \begin{cases} \vartheta_{\boldsymbol{a}} \log |z|^2 & \text{if } |z| < \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}, \\ \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \leq |z| \leq \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ \theta_{\boldsymbol{a}} \log |z|^2 & \text{if } |z| > \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \end{cases}$$

then the positive part of $\overline{D}_{\boldsymbol{a}}$ is $\overline{P}_{\boldsymbol{a}} = (P_{\boldsymbol{a}}, p_{\boldsymbol{a}})$, where $\sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}$ is treated as ∞ if $\theta_{\boldsymbol{a}} = 1$.

Proof. First we consider the case where $\overline{D}_{\boldsymbol{a}}$ is big, that is, $a_0 + a_1 > 1$ by Theorem 2.3. In this case, $0 \leq \vartheta_{\boldsymbol{a}} < \theta_{\boldsymbol{a}} \leq 1$. The existence of the Zariski decomposition follows from [9, Theorem 9.2.1]. Here we consider functions

$$r_1 : \left\{ z \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; |z| < \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}} \right\} \to \mathbb{R}$$

and

$$r_2 : \left\{ z \in \mathbb{P}^1(\mathbb{C}) \;\middle|\; |z| > \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \right\} \to \mathbb{R}$$

given by

$$r_1(z) = \begin{cases} 0 & \text{if } |z| < \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}, \\ -\vartheta_{\boldsymbol{a}} \log |z|^2 + \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \leq |z| < \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

and

$$r_2(z) = \begin{cases} -\theta_{\boldsymbol{a}} \log|z|^2 + \log(a_0 + a_1|z|^2) & \text{if } \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} < |z| \leq \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ 0 & \text{if } |z| > \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

In order to see that $p_{\boldsymbol{a}}$ is a $P_{\boldsymbol{a}}$-Green function of $(C^0 \cap \mathrm{PSH})$-type on $\mathbb{P}^1(\mathbb{C})$, it is sufficient to check that r_1 and r_2 are continuous and subharmonic on each area. Let us see that r_1 is continuous and subharmonic. If $\vartheta_{\boldsymbol{a}} = 0$, then the assertion is obvious, so that we may assume that $\vartheta_{\boldsymbol{a}} > 0$. First of all, as $\varphi_{\boldsymbol{a}}(1-\vartheta_{\boldsymbol{a}}, \vartheta_{\boldsymbol{a}}) = 0$, we have $r_1(z) = 0$ if $|z| = \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}$, and hence r_1 is continuous. It is obvious that r_1 is subharmonic on

$$\left\{ z \in \mathbb{C} \;\middle|\; |z| < \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \right\} \cup \left\{ z \in \mathbb{C} \;\middle|\; \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} < |z| < \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}} \right\}.$$

By using Claim 1.2.1,

$$\begin{aligned} \vartheta_{\boldsymbol{a}} \log|z|^2 &= (1-\vartheta_{\boldsymbol{a}})\log(1) + \vartheta_{\boldsymbol{a}} \log|z|^2 \\ &\leq \log(a_0 + a_1|z|^2) + \varphi_{\boldsymbol{a}}(1-\vartheta_{\boldsymbol{a}}, \vartheta_{\boldsymbol{a}}) = \log(a_0 + a_1|z|^2). \end{aligned}$$

Thus $r_1 \geq 0$. Therefore, if $|z| = \sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}$, then

$$r_1(z) = 0 \leq \frac{1}{2\pi} \int_0^{2\pi} r_1(z + \epsilon e^{\sqrt{-1}t}) dt$$

for a small positive real number ϵ, and hence r_1 is subharmonic. In the similar way, we can check that r_2 is continuous and subharmonic.

Next let us see that $\overline{P}_{\boldsymbol{a}}$ is nef. As $r_1(0) = 0$ and $r_2(\infty) = 0$, we have

$$\widehat{\deg}(\overline{P}_{\boldsymbol{a}}\big|_{H_0}) = \widehat{\deg}(\overline{P}_{\boldsymbol{a}}\big|_{H_1}) = 0.$$

Note that

$$\overline{P}_{\boldsymbol{a}} + \vartheta_{\boldsymbol{a}}\widehat{(z)} = ((\theta_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}})H_0, p_{\boldsymbol{a}}(z) - \vartheta_{\boldsymbol{a}} \log|z|^2)$$

and

$$p_{\boldsymbol{a}}(z) - \vartheta_{\boldsymbol{a}} \log|z|^2 = \begin{cases} r_1(z) & \text{if } |z| \leq \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ (\theta_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}}) \log|z|^2 & \text{if } |z| > \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

Therefore, $p_{\boldsymbol{a}}(z) - \vartheta_{\boldsymbol{a}} \log|z|^2 \geq 0$ on $\mathbb{P}^1(\mathbb{C})$, which means that $\overline{P}_{\boldsymbol{a}} + \vartheta_{\boldsymbol{a}}\widehat{(z)}$ is effective. Let γ be a 1-dimensional closed integral subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ with $\gamma \neq H_0, H_1$. Then

$$\widehat{\deg}(\overline{P}_{\boldsymbol{a}}\big|_{\gamma}) = \widehat{\deg}(((\theta_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}})H_0, p_{\boldsymbol{a}} - \vartheta_{\boldsymbol{a}} \log|z|^2)\big|_{\gamma}) \geq 0.$$

By using Proposition 1.5, we have $\mu_{H_0}(\overline{D}_{\boldsymbol{a}}) = 1 - \theta_{\boldsymbol{a}}$ and $\mu_{H_1}(\overline{D}_{\boldsymbol{a}}) = \vartheta_{\boldsymbol{a}}$. Thus the positive part of $\overline{D}_{\boldsymbol{a}}$ can be written by a form $(P_{\boldsymbol{a}}, q)$, where q is a $P_{\boldsymbol{a}}$-Green function of $(C^0 \cap \mathrm{PSH})$-type on $\mathbb{P}^1(\mathbb{C})$ (cf. [9, Claim 9.3.5.1 and Proposition 9.3.1]). Note that $\overline{P}_{\boldsymbol{a}}$ is nef and $\overline{P}_{\boldsymbol{a}} \leq \overline{D}_{\boldsymbol{a}}$, so that

$$p_{\boldsymbol{a}}(z) \leq q(z) \leq \log(a_0 + a_1|z|^2).$$

We choose a continuous function u such that $p_{\boldsymbol{a}} + u = q$. Then $u(z) = 0$ on

$$\sqrt{\frac{a_0\vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}} \leq |z| \leq \sqrt{\frac{a_0\theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}.$$

Moreover, since $q(z) = \vartheta_{\boldsymbol{a}} \log|z|^2 + u(z)$ on $|z| \leq \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}$, u is subharmonic on $|z| \leq \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}$. On the other hand, $u(0) = 0$ because

$$\widehat{\deg}(\left.(P_{\boldsymbol{a}}, q)\right|_{H_1}) = u(0) = 0.$$

Therefore, $u = 0$ on $|z| \leq \sqrt{\frac{a_0 \vartheta_{\boldsymbol{a}}}{a_1(1-\vartheta_{\boldsymbol{a}})}}$ by the maximal principle. In a similar way, we can see that $u = 0$ on $|z| \geq \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}$.

Next we consider the case where $a_0 + a_1 = 1$. By Claim 1.2.1,

$$a_1 \log|z|^2 \leq \log(a_0 + a_1|z|^2)$$

on $\mathbb{P}^1(\mathbb{C})$. Thus $-a_1\widehat{(z)} \leq \overline{D}_{\boldsymbol{a}}$, and hence the Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ exists by [9, Theorem 9.2.1]. Let $\overline{P}$ be the positive part of $\overline{D}_{\boldsymbol{a}}$. Then $-a_1\widehat{(z)} \leq \overline{P}$.

Let us consider the converse inequality. Let t be a real number with $t > 1$. Since $\overline{P} \leq \overline{D}_{\boldsymbol{a}} \leq \overline{D}_{t\boldsymbol{a}}$, we have $\overline{P} \leq \overline{P}_{t\boldsymbol{a}}$ because $\overline{P}_{t\boldsymbol{a}}$ is the positive part of $\overline{D}_{t\boldsymbol{a}}$ by the previous observation. Since $\varphi_{t\boldsymbol{a}} = \varphi_{\boldsymbol{a}} + \log(t)$, we have $\lim_{t\to 1} \vartheta_{t\boldsymbol{a}} = \lim_{t\to 1} \theta_{t\boldsymbol{a}} = a_1$. Therefore, we can see

$$\lim_{t\to 1} \overline{P}_{t\boldsymbol{a}} = \overline{P}_{\boldsymbol{a}} = -a_1\widehat{(z)}.$$

Thus $\overline{P} \leq -a_1\widehat{(z)}$.

Finally we consider the case where $a_0 + a_1 < 1$. Then, by Theorem 2.3, $\overline{D}_{\boldsymbol{a}}$ is not pseudo-effective. Thus the Zariski decomposition does not exist by [9, Proposition 9.3.2]. □

5. Weak Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$

Let X be a d-dimensional, projective, generically smooth and normal arithmetic variety. Let $\overline{D}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on X. A decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a *weak Zariski decomposition of* $\overline{D}$ if the following conditions are satisfied:

(1) $\overline{P}$ is a nef and big arithmetic $\mathbb{R}$-divisor of $(C^0 \cap \mathrm{PSH})$-type.
(2) $\overline{N}$ is an effective arithmetic $\mathbb{R}$-divisor of C^0-type.
(3) $\mathrm{mult}_\Gamma(N) \leq \mu_\Gamma(\overline{D})$ for any horizontal prime divisor Γ on X, that is, Γ is a reduced and irreducible divisor Γ on X such that Γ is flat over $\mathbb{Z}$.

Note that the Zariski decomposition of a big arithmetic $\mathbb{R}$-divisor of C^0-type on an arithmetic surface is a weak Zariski decomposition (cf. [9, Claim 9.3.5.1]). The above property (3) implies that $\mathrm{mult}_\Gamma(N) = \mu_\Gamma(\overline{D})$ for any horizontal prime divisor Γ on X. Indeed, by (2) and (5) in Proposition 3.1,

$$\mu_\Gamma(\overline{D}) \leq \mu_\Gamma(\overline{P}) + \mathrm{mult}_\Gamma(N) = \mathrm{mult}_\Gamma(N) \leq \mu_\Gamma(\overline{D}).$$

From now on, we use the same notation as in Section 1. Let us begin with the following lemma.

Lemma 5.1. *Let $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ and $g : Y \to X$ be birational morphisms of projective, generically smooth and normal arithmetic varieties. If $f^*(\overline{D}_{\boldsymbol{a}})$ admits a weak Zariski decomposition, then $g^*(f^*(\overline{D}_{\boldsymbol{a}}))$ also admits a weak Zariski decomposition.*

Proof. Let $f^*(\overline{D}_{\boldsymbol{a}}) = \overline{P} + \overline{N}$ be a weak Zariski decomposition of $f^*(\overline{D}_{\boldsymbol{a}})$. We denote birational morphisms $X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$ and $Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ by $f_{\mathbb{Q}}$ and $g_{\mathbb{Q}}$ respectively. We set

$$\widetilde{\Theta}_{\boldsymbol{a}} = \{\widetilde{e} \in \mathbb{R}^{n+1} \mid e \in \Theta_{\boldsymbol{a}}\},$$

$f^*_{\mathbb{Q}}(H_i) = \sum_j a_{ij} D_j$ for $i = 0, \ldots, n$ and $N = \sum_j b_j D_j$ on $X_{\mathbb{Q}}$, where D_j's are reduced and irreducible divisors on $X_{\mathbb{Q}}$. Since

$$lH_0 + (z^{\boldsymbol{e}}) = (l - \boldsymbol{e}(1) - \cdots - \boldsymbol{e}(n))H_0 + \boldsymbol{e}(1)H_1 + \cdots + \boldsymbol{e}(n)H_n$$

for $\boldsymbol{e} \in l\Theta_{\boldsymbol{a}} \cap \mathbb{Z}^n$, by Lemma 3.2, we have

$$\mu_{D_j}(f^*(\overline{D}_{\boldsymbol{a}})) = \min\left\{ \sum_{i=0}^n x_i a_{ij} \;\middle|\; (x_0, \ldots, x_n) \in \widetilde{\Theta}_{\boldsymbol{a}} \right\}.$$

Thus

$$b_j \leq \min\left\{ \sum_{i=0}^n x_i a_{ij} \;\middle|\; (x_0, \ldots, x_n) \in \widetilde{\Theta}_{\boldsymbol{a}} \right\}.$$

for all j.

Here let us see that $g^*(f^*(\overline{D}_{\boldsymbol{a}})) = g^*(\overline{P}) + g^*(\overline{N})$ is a weak Zariski decomposition. For this purpose, it is sufficient to see that $\mathrm{mult}_\Gamma(g^*(N)) \leq \mu_\Gamma(g^*(f^*(\overline{D}_{\boldsymbol{a}})))$ for any horizontal prime divisor Γ on Y. If we set $c_j = \mathrm{mult}_\Gamma(g^*_{\mathbb{Q}}(D_j))$, then

$$d_i := \mathrm{mult}_\Gamma(g^*_{\mathbb{Q}}(f^*_{\mathbb{Q}}(H_i))) = \sum_j a_{ij} c_j.$$

For $(x_0, \ldots, x_n) \in \widetilde{\Theta}_{\boldsymbol{a}}$,

$$\sum_i x_i d_i = \sum_j \left(\sum_i x_i a_{ij} \right) c_j \geq \sum_j b_j c_j = \mathrm{mult}_\Gamma(g^*_{\mathbb{Q}}(N)),$$

which yields $\mu_\Gamma(g^*(f^*(\overline{D}_{\boldsymbol{a}}))) \geq \mathrm{mult}_\Gamma(g^*(N))$. □

Next let us consider the following lemma:

Lemma 5.2. *Let Θ be a compact convex set in $\mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection given by $p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$. Then $p(\Theta)$ is a compact convex set in $\mathbb{R}^{n-1}$ and there exist a concave function θ on $p(\Theta)$ and a convex function ϑ on $p(\Theta)$ such that*

$$\Theta = \left\{ (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (x_1, \ldots, x_{n-1}) \in p(\Theta), \\ \vartheta(x_1, \ldots, x_{n-1}) \leq x_n \leq \theta(x_1, \ldots, x_{n-1}) \end{array} \right\}.$$

Proof. Obviously $p(\Theta)$ is a compact convex set in $\mathbb{R}^{n-1}$. For $(x_1, \ldots, x_{n-1}) \in p(\Theta)$, we set

$$\begin{cases} \theta(x_1, \ldots, x_{n-1}) := \max\{x_n \in \mathbb{R} \mid (x_1, \ldots, x_{n-1}, x_n) \in \Theta\}, \\ \vartheta(x_1, \ldots, x_{n-1}) := \min\{x_n \in \mathbb{R} \mid (x_1, \ldots, x_{n-1}, x_n) \in \Theta\}. \end{cases}$$

Clearly

$$\Theta = \left\{ (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{l} (x_1, \ldots, x_{n-1}) \in p(\Theta), \\ \vartheta(x_1, \ldots, x_{n-1}) \leq x_n \leq \theta(x_1, \ldots, x_{n-1}) \end{array} \right\}.$$

We need to show that θ (resp. ϑ) is a concave (resp. convex) function. Since

$$(x_1, \ldots, x_{n-1}, \theta(x_1, \ldots, x_{n-1})), (x'_1, \ldots, x'_{n-1}, \theta(x'_1, \ldots, x'_{n-1})) \in \Theta$$

for $(x_1, \ldots, x_{n-1}), (x'_1, \ldots, x'_{n-1}) \in p(\Theta)$, we have

$$\lambda(x_1, \ldots, x_{n-1}, \theta(x_1, \ldots, x_{n-1})) + (1 - \lambda)(x'_1, \ldots, x'_{n-1}, \theta(x'_1, \ldots, x'_{n-1})) \in \Theta$$

for $0 \leq \lambda \leq 1$, which shows that

$$\lambda\theta(x_1, \dots, x_{n-1}) + (1-\lambda)\theta(x'_1, \dots, x'_{n-1}) \leq \theta(\lambda(x_1, \dots, x_{n-1}) + (1-\lambda)(x'_1, \dots, x'_{n-1})).$$

Thus θ is concave. Similarly we can see that ϑ is convex. $\square$

Remark 5.3. If $p(\Theta)$ is a polytope in Lemma 5.2, then θ and ϑ are continuous on $p(\Theta)$ (cf. [3]). In general, θ and ϑ are not necessarily continuous on $p(\Theta)$. Indeed, let us consider the following set:

$$\Theta = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq y \leq 1,\ 0 \leq z \leq 1,\ x^2 \leq yz\}.$$

Since

$$x^2 \leq yz \iff x^2 + \left(\frac{y-z}{2}\right)^2 \leq \left(\frac{y+z}{2}\right)^2,$$

we can easily see that Θ is a compact convex set in $\mathbb{R}^3$. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection given by $p(x, y, z) = (x, y)$. Then

$$p(\Theta) = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq y \leq 1\}.$$

Moreover, ϑ is given by

$$\vartheta(x, y) = \begin{cases} x^2/y & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

and hence ϑ is not continuous at $(0, 0)$.

Note that $\Theta_{\boldsymbol{a}}$ is a compact convex set of $\mathbb{R}^n$. We say a hyperplane $\alpha_1 x_1 + \cdots + \alpha_n x_n = \beta$ in $\mathbb{R}^n$ is a *supporting hyperplane* of $\Theta_{\boldsymbol{a}}$ at $(b_1, \dots, b_n) \in \Theta_{\boldsymbol{a}}$ if

$$\Theta_{\boldsymbol{a}} \subseteq \{\alpha_1 x_1 + \cdots + \alpha_n x_n \geq \beta\} \quad \text{and} \quad \alpha_1 b_1 + \cdots + \alpha_n b_n = \beta.$$

Proposition 5.4. *Let $(b_1, \dots, b_n) \in \partial(\Theta_{\boldsymbol{a}})$, that is, $(b_1, \dots, b_n)$ is a boundary point of $\Theta_{\boldsymbol{a}}$. We set $b_0 = 1 - b_1 - \cdots - b_n$. We assume*

$$a_0 + a_1 + \cdots + a_n > 1 \quad \textit{and} \quad \#\{i \mid 0 \leq i \leq n,\ b_i = 0\} \leq 1.$$

Then $\Theta_{\boldsymbol{a}}$ has a unique supporting hyperplane at $(b_1, \dots, b_n)$. Moreover, in the case where $b_i = 0$, the supporting hyperplane is given by

$$\begin{cases} x_1 + \cdots + x_n = 1 & \textit{if } b_0 = 0, \\ x_i = 0 & \textit{if } b_i = 0 \textit{ for some } i \textit{ with } 1 \leq i \leq n. \end{cases}$$

Proof. Here we set

$$\phi_{\boldsymbol{a}}(x_1, \dots, x_n) = \varphi_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \dots, x_n)$$

on $\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\}$. Then

$$\Theta_{\boldsymbol{a}} = \{(x_1, \dots, x_n) \in \Delta_n \mid \phi_{\boldsymbol{a}}(x_1, \dots, x_n) \geq 0\}.$$

First we assume that $(b_1, \dots, b_n) \notin \partial(\Delta_n)$. Then $\phi_{\boldsymbol{a}}(b_1, \dots, b_n) = 0$. Note that, for $(x_1, \dots, x_n) \in \Delta_n \setminus \partial(\Delta_n)$,

$$(\phi_{\boldsymbol{a}})_{x_1}(x_1, \dots, x_n) = \cdots = (\phi_{\boldsymbol{a}})_{x_n}(x_1, \dots, x_n) = 0 \iff (x_1, \dots, x_n) = \left(\frac{a_1}{a_0 + \cdots + a_n}, \dots, \frac{a_n}{a_0 + \cdots + a_n}\right),$$

and $\phi_{\boldsymbol{a}}\left(\frac{a_1}{a_0+\cdots+a_n}, \dots, \frac{a_n}{a_0+\cdots+a_n}\right) = \log(a_0+\cdots+a_n) > 0$. Thus we have

$$((\phi_{\boldsymbol{a}})_{x_1}(b_1,\dots,b_n),\dots,(\phi_{\boldsymbol{a}})_{x_1}(b_1,\dots,b_n)) \neq (0,\dots,0),$$

which means that $\Theta_{\boldsymbol{a}}$ has a unique supporting hyperplane at $(b_1,\dots,b_n)$.

Next we assume that $(b_1,\dots,b_n) \in \partial(\Delta_n)$. Considering the following linear transformations:

$$\begin{cases} x'_1 = x_1, \\ \ \vdots \quad \vdots \\ x'_{n-1} = x_{n-1}, \\ x'_n = 1 - x_1 - \cdots - x_n, \end{cases} \qquad \begin{cases} x'_1 = x_1, \\ \ \vdots \quad \vdots \\ x'_i = x_n, \\ \ \vdots \quad \vdots \\ x'_n = x_i, \end{cases}$$

we may assume $b_n = 0$. Note that $(b_1,\dots,b_{n-1}) \in \Delta_{n-1} \setminus \partial(\Delta_{n-1})$. Let $p : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection given by $p(x_1,\dots,x_n) = (x_1,\dots,x_{n-1})$. By Lemma 5.2, there are a concave function θ on $p(\Theta_{\boldsymbol{a}})$ and a convex function ϑ on $p(\Theta_{\boldsymbol{a}})$ such that

$$\Theta_{\boldsymbol{a}} = \left\{ (x_1,\dots,x_{n-1},x_n) \ \middle|\ \begin{array}{l} (x_1,\dots,x_{n-1}) \in p(\Theta_{\boldsymbol{a}}), \\ \vartheta(x_1,\dots,x_{n-1}) \leq x_n \leq \theta(x_1,\dots,x_{n-1}) \end{array} \right\}.$$

Claim 5.4.1. $(b_1,\dots,b_{n-1})$ *is an interior point of* $p(\Theta_{\boldsymbol{a}})$. *In particular,* ϑ *is continuous around* $(b_1,\dots,b_{n-1})$ (*cf.* [5, Theorem 2.2]).

Proof. Let us consider a function $\psi : [0, 1-b_1-\cdots-b_{n-1}] \to \mathbb{R}$ given by $\psi(t) = \phi_{\boldsymbol{a}}(b_1,\dots,b_{n-1},t)$. Note that

$$\psi'(t) = \log\frac{a_n}{a_0}\left(\frac{1-b_1-\cdots-b_{n-1}}{t} - 1\right) > 0$$

on $\left(0, \frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n}\right)$. Thus

$$\phi_{\boldsymbol{a}}\left(b_1,\dots,b_{n-1},\frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n}\right) > \phi_{\boldsymbol{a}}(b_1,\dots,b_{n-1},0) \geq 0.$$

Therefore, as $\left(b_1,\dots,b_{n-1},\frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n}\right) \in \Delta_n \setminus \partial(\Delta_n)$, we can find a sufficiently small positive number ϵ such that

$$\prod_{i=1}^{n-1}(b_i-\epsilon, b_i+\epsilon) \times \left(\frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n} - \epsilon, \frac{a_n(1-b_1-\cdots-b_{n-1})}{a_0+a_n} + \epsilon\right)$$

is a subset of $\Theta_{\boldsymbol{a}}$, and hence

$$(b_1,\dots,b_{n-1}) \in \prod_{i=1}^{n-1}(b_i-\epsilon, b_i+\epsilon) \subseteq p(\Theta_{\boldsymbol{a}}).$$

$\square$

We set $\boldsymbol{a}' = (a_0,\dots,a_{n-1})$. Then

$$\Theta_{\boldsymbol{a}'} = \{(x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1,\dots,x_{n-1},0) \in \Theta_{\boldsymbol{a}}\}.$$

Clearly $(b_1,\dots,b_{n-1}) \in \Theta_{\boldsymbol{a}'}$ and $\vartheta \equiv 0$ on $\Theta_{\boldsymbol{a}'}$.

Claim 5.4.2. *ϑ is a continuously differentiable function around $(b_1, \dots, b_{n-1})$ such that*

$$\vartheta_{x_1}(b_1, \dots, b_{n-1}) = \cdots = \vartheta_{x_{n-1}}(b_1, \dots, b_{n-1}) = 0.$$

Proof. By Claim 5.4.1, there is a positive number ϵ such that

$$b_1 - \epsilon > 0, \dots, b_{n-1} - \epsilon > 0,\ (b_1 + \epsilon) + \cdots + (b_{n-1} + \epsilon) < 1$$

and ϑ is continuous on $U = \prod_{i=1}^{n-1}(b_i - \epsilon, b_i + \epsilon)$. If $(x_1, \dots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}$, then $\vartheta(x_1, \dots, x_{n-1}) > 0$, and hence

$$\phi_{\boldsymbol{a}}(x_1, \dots, x_{n-1}, \vartheta(x_1, \dots, x_{n-1})) = 0$$

for $(x_1, \dots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}$. Note that

$$(\phi_{\boldsymbol{a}})_{x_i} = \log \frac{a_i}{a_0} \left(\frac{1 - x_1 - \cdots - x_n}{x_i} \right). \tag{5.4.3}$$

Since $\vartheta(b_1, \dots, b_{n-1}) = 0$ and ϑ is continuous at $(b_1, \dots, b_{n-1})$, choosing a smaller ϵ if necessarily, we may assume that

$$(\phi_{\boldsymbol{a}})_{x_n}(x_1, \dots, x_{n-1}, \vartheta(x_1, \dots, x_{n-1})) > 0$$

for all $(x_1, \dots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}$. Thus, by using the implicit function theorem, ϑ is a C^∞ function on $U \setminus \Theta_{\boldsymbol{a}'}$ and

$$\vartheta_{x_i}(x_1, \dots, x_{n-1}) = -\frac{(\phi_{\boldsymbol{a}})_{x_i}(x_1, \dots, x_{n-1}, \vartheta(x_1, \dots, x_{n-1}))}{(\phi_{\boldsymbol{a}})_{x_n}(x_1, \dots, x_{n-1}, \vartheta(x_1, \dots, x_{n-1}))}. \tag{5.4.4}$$

Let us consider a function γ_i on U given by

$$\gamma_i(x_1, \dots, x_{n-1}) = \begin{cases} 0 & \text{if } (x_1, \dots, x_{n-1}) \in U \cap \Theta_{\boldsymbol{a}'}, \\ \vartheta_{x_i}(x_1, \dots, x_{n-1}) & \text{if } (x_1, \dots, x_{n-1}) \in U \setminus \Theta_{\boldsymbol{a}'}. \end{cases}$$

Then, by using (5.4.3) and (5.4.4), it is easy to see that γ_i is continuous on U. Thus the claim follows. □

The above claim shows that $\Theta_{\boldsymbol{a}}$ has the unique supporting hyperplane at $(b_1, \dots, b_n)$ and it is given by $x_n = 0$. □

Corollary 5.5. *We assume that $a_0 < 1$ and $a_0 + a_1 + \cdots + a_n \geq 1$. Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}_{>0}$ and $(b_1, \dots, b_n) \in \Theta_{\boldsymbol{a}}$ such that*

$$\alpha_1 b_1 + \cdots + \alpha_n b_n = \min\{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid (x_1, \dots, x_n) \in \Theta_{\boldsymbol{a}}\}.$$

Then $(b_1, \dots, b_n) \notin \partial(\Delta_n)$.

Proof. We prove it by induction on n. If $n = 1$, then the assertion is obvious, so that we may assume $n > 1$. If $a_0 + \cdots + a_n = 1$, then

$$\Theta_{\boldsymbol{a}} = \left\{ \left(\frac{a_1}{a_0 + \cdots + a_n}, \dots, \frac{a_n}{a_0 + \cdots + a_n} \right) \right\}.$$

In this case, the assertion is also obvious. Thus we may assume that $a_0 + \cdots + a_n > 1$.

We assume that $b_i = 0$ for some $1 \leq i \leq n$. Then, since $\Theta_{\boldsymbol{a}} \cap \{x_i = 0\} \neq \emptyset$, we have

$$a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_n \geq 1.$$

Thus, by the hypothesis of induction,

$$b_1 \neq 0, \dots, b_{i-1} \neq 0, b_{i+1} \neq 0, \dots, b_n \neq 0, b_1 + \cdots + b_n \neq 1.$$

Therefore, by Proposition 5.4, we have the unique supporting hyperplane $x_i = 0$ of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n)$. On the other hand, $\alpha_1 x_1 + \cdots + \alpha_n x_n = \alpha_1 b_1 + \cdots + \alpha_n b_n$ is also a supporting hyperplane of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n)$. This is a contradiction.

Next we assume that $b_1 + \cdots + b_n = 1$. Since $b_i \neq 0$ for all i, by Proposition 5.4, the unique supporting hyperplane of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n)$ is $x_1 + \cdots + x_n = 1$, which yields $\alpha_1 = \cdots = \alpha_n$, and hence $\Theta_{\boldsymbol{a}} \subseteq \{x_1 + \cdots + x_n = 1\}$. This is a contradiction because

$$\left(\frac{a_1}{a_0 + \cdots + a_n}, \ldots, \frac{a_n}{a_0 + \cdots + a_n}\right) \in \Theta_{\boldsymbol{a}},$$

as required. □

Theorem 5.6. *We assume that $n \geq 2$ and $\overline{D}_{\boldsymbol{a}}$ is big. Then $\overline{D}_{\boldsymbol{a}}$ is nef if and only if there is a birational morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth and normal arithmetic varieties such that $f^*(\overline{D}_{\boldsymbol{a}})$ admits a weak Zariski decomposition on X.*

Proof. If $\overline{D}_{\boldsymbol{a}}$ is nef, then $\overline{D}_{\boldsymbol{a}} = \overline{D}_{\boldsymbol{a}} + (0, 0)$ is a weak Zariski decomposition. Next we assume that $\overline{D}_{\boldsymbol{a}}$ is not nef and there is a birational morphism $f : X \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth and normal arithmetic varieties such that $f^*(\overline{D}_{\boldsymbol{a}})$ admits a weak Zariski decomposition $f^*(\overline{D}_{\boldsymbol{a}}) = \overline{P} + \overline{N}$ on X. By our assumptions, $a_0 + \cdots + a_n > 1$ and $a_i < 1$ for some i. Renumbering the homogeneous coordinate $T_0, \ldots, T_n$, we may assume $a_0 < 1$. Let ξ be the generic point of $H_1 \cap \cdots \cap H_n$, that is, $\xi = (1 : 0 : \cdots : 0) \in \mathbb{P}^n(\mathbb{Q})$. Let L_i be the strict transform of H_i by f for $i = 0, \ldots, n$. We denote the birational morphism $X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$ by $f_{\mathbb{Q}}$. Let $f' : X' \to \mathbb{P}^n_{\mathbb{Z}}$ be the blowing-up along $H_1 \cap \cdots \cap H_n$. By using Lemma 5.1 and [7], we may assume the following:

(1) Let Σ be the exceptional set of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$. Then Σ is a divisor on $X_{\mathbb{Q}}$ and $(\Sigma + (L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}})_{\mathrm{red}}$ is a normal crossing divisor on $X_{\mathbb{Q}}$.
(2) There is a birational morphism $g : X \to X'$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[dr, "g"] \arrow[dd, "f"'] & \\ & X' \arrow[dl, "f'"] \\ \mathbb{P}^n_{\mathbb{Z}} & \end{tikzcd}$$

Claim 5.6.1. *There are $\xi' \in X(\mathbb{Q})$ and a reduced and irreducible divisor E on $X_{\mathbb{Q}}$ with the following properties:*

(a) *$f_{\mathbb{Q}}(\xi') = \xi$ and $\xi' \in E \cap (L_n)_{\mathbb{Q}}$.*
(b) *E and $(L_n)_{\mathbb{Q}}$ is non-singular at ξ'.*
(c) *E is exceptional with respect to $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$.*
(d) *There are positive integers $\alpha_1, \ldots, \alpha_n$ such that*

$$f^*_{\mathbb{Q}}(H_i) = \alpha_i E + (\textit{the sum of divisors which do not pass through } \xi')$$

for $i = 1, \ldots, n-1$ and

$$f^*_{\mathbb{Q}}(H_n) = (L_n)_{\mathbb{Q}} + \alpha_n E + (\textit{the sum of divisors which do not pass through } \xi').$$

Proof. Let L'_n be the strict transform of H_n by f' and Σ' the exceptional set of $f'_{\mathbb{Q}} : X'_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$. Then $\Sigma' = \mathbb{P}^{n-1}_{\mathbb{Q}}$ and $D' := (L'_n)_{\mathbb{Q}} \cap \Sigma' = \mathbb{P}^{n-2}_{\mathbb{Q}}$. Let $h : L_n \to L'_n$ and $h_{\mathbb{Q}}$:

$(L_n)_{\mathbb{Q}} \to (L'_n)_{\mathbb{Q}}$ be the birational morphisms induced by $g : X \to X'$ and $g_{\mathbb{Q}} : X_{\mathbb{Q}} \to X'_{\mathbb{Q}}$ respectively. Let D be the strict transformation of D' by $h_{\mathbb{Q}}$. As before, let Σ be the exceptional set of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$. Let

$$(\Sigma + (L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}})_{\mathrm{red}} = (L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}} + E_0 + \cdots + E_l$$

be the irreducible decomposition such that E_i's are exceptional with respect to $f_{\mathbb{Q}}$. Since $D \subseteq (L_n)_{\mathbb{Q}} \cap \Sigma$, there is E_i such that $D \subseteq (L_n)_{\mathbb{Q}} \cap E_i$. Renumbering $E_0, \ldots, E_l$, we may assume that $E_i = E_l$. As $(L_0)_{\mathbb{Q}} + \cdots + (L_n)_{\mathbb{Q}} + E_0 + \cdots + E_l$ is a normal crossing divisor on $X_{\mathbb{Q}}$, we have

$$\begin{cases} D \cap \operatorname{Sing}((L_n)_{\mathbb{Q}}) \subsetneq D,\ D \cap \operatorname{Sing}(E) \subsetneq D, \\ D \cap (L_i)_{\mathbb{Q}} \subsetneq D \ (i = 0, \ldots, n-1), \\ D \cap E_j \subsetneq D \ (j = 0, \ldots, l-1). \end{cases}$$

Note that $D(\mathbb{Q})$ is dense in D because $D \to D'$ is birational. Thus we can find $\xi' \in D(\mathbb{Q})$ such that

$$\xi' \notin (D \cap \operatorname{Sing}((L_n)_{\mathbb{Q}})) \cup (D \cap \operatorname{Sing}(E)) \cup \bigcup_{i=0}^{n-1} (D \cap (L_i)_{\mathbb{Q}}) \cup \bigcup_{j=0}^{l-1} (D \cap E_j).$$

Therefore the claim follows. $\square$

Note that

$$\begin{aligned} f^*_{\mathbb{Q}}(lH_0 + (z_1^{e_1} \cdots z_n^{e_n})) &= f^*_{\mathbb{Q}}((l - e_1 - \cdots - e_n)H_0 + e_1 H_1 + \cdots + e_n H_n) \\ &= e_n (L_n)_{\mathbb{Q}} + (\alpha_1 e_1 + \cdots + \alpha_n e_n) E \\ &\qquad + (\text{the sum of divisors which do not pass through } \xi'). \end{aligned}$$

Therefore, by Lemma 3.2,

$$\begin{cases} \mu_{\xi'}(f^*(\overline{D}_{\boldsymbol{a}})) = \min\{\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + (\alpha_n + 1) x_n \mid (x_1, \cdots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ \mu_E(f^*(\overline{D}_{\boldsymbol{a}})) = \min\{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid (x_1, \cdots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ \mu_{L_n}(f^*(\overline{D}_{\boldsymbol{a}})) = \min\{x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}. \end{cases}$$

Further,

$$\operatorname{mult}_{\xi'}(N) = \operatorname{mult}_E(N) + \operatorname{mult}_{L_n}(N) \le \mu_E(f^*(\overline{D}_{\boldsymbol{a}})) + \mu_{L_n}(f^*(\overline{D}_{\boldsymbol{a}})).$$

By (2) and (5) in Proposition 3.1,

$$0 = \mu_{\xi'}(\overline{P}) \ge \mu_{\xi'}(f^*(\overline{D}_{\boldsymbol{a}})) - \operatorname{mult}_{\xi'}(N).$$

Therefore, if we set

$$\begin{cases} A = \min\{\alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1} + (\alpha_n + 1) x_n \mid (x_1, \cdots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ B = \min\{\alpha_1 x_1 + \cdots + \alpha_n x_n \mid (x_1, \cdots, x_n) \in \Theta_{\boldsymbol{a}}\}, \\ C = \min\{x_n \mid (x_1, \ldots, x_n) \in \Theta_{\boldsymbol{a}}\}, \end{cases}$$

then we have $0 \ge A - B - C$. We choose $(b_1, \ldots, b_n) \in \Theta_{\boldsymbol{a}}$ such that

$$A = \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1} + (\alpha_n + 1) b_n.$$

Thus, as $\alpha_1 b_1 + \cdots + \alpha_n b_n \ge B$ and $b_n \ge C$, we have

$$\begin{aligned} 0 &\ge A - B - C \\ &\ge \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1} + (\alpha_n + 1) b_n - (\alpha_1 b_1 + \cdots + \alpha_n b_n) - b_n = 0, \end{aligned}$$

which implies $\alpha_1 b_1 + \cdots + \alpha_n b_n = B$ and $b_n = C$. On the other hand, by Corollary 5.5, $(b_1, \ldots, b_n) \not\in \partial(\Delta_n)$, and hence there is a unique supporting hyperplane of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n)$ by Proposition 5.4. This is a contradiction because

$$\begin{cases} \alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1} + (\alpha_n + 1)x_n = A, \\ \alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1} + \alpha_n x_n = B, \\ x_n = C \end{cases}$$

are distinct supporting hyperplanes of $\Theta_{\boldsymbol{a}}$ at $(b_1, \ldots, b_n)$. □

6. Fujita's approximation of $\overline{D}_{\boldsymbol{a}}$

Fujita's approximation of arithmetic divisors has established by Chen and Yuan (cf. [2], [10], [8] and [9]). In this section, we consider Fujita's approximation of $\overline{D}_{\boldsymbol{a}}$ in terms of rational interior points of $\Theta_{\boldsymbol{a}}$.

First of all, we fix notation. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in \mathbb{R}^n$ and $\phi_1, \ldots, \phi_r \in \mathbb{R}$. We define a function $\phi_{(\boldsymbol{x}_1,\phi_1),\ldots,(\boldsymbol{x}_r,\phi_r)}$ on $\Theta = \mathrm{Conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$ to be

$$\phi_{(\boldsymbol{x}_1,\phi_1),\ldots,(\boldsymbol{x}_r,\phi_r)}(\boldsymbol{x}) := \max\left\{ \sum_{i=1}^r \lambda_i \phi_i \;\middle|\; \begin{array}{l} \boldsymbol{x} = \sum_{i=1}^r \lambda_i \boldsymbol{x}_i, \\ \lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0},\ \sum_{i=1}^r \lambda_i = 1 \end{array} \right\}.$$

In other words, $\phi_{(\boldsymbol{x}_1,\phi_1),\ldots,(\boldsymbol{x}_r,\phi_r)}$ is given by

$$\phi_{(\boldsymbol{x}_1,\phi_1),\ldots,(\boldsymbol{x}_r,\phi_r)}(\boldsymbol{x}) = \max\{\phi \in \mathbb{R} \mid (\boldsymbol{x}, \phi) \in \mathrm{Conv}\{(\boldsymbol{x}_1, \phi_1), \ldots, (\boldsymbol{x}_r, \phi_r)\} \subseteq \mathbb{R}^n \times \mathbb{R}\}.$$

Thus we can easily see that $\phi_{(\boldsymbol{x}_1,\phi_1),\ldots,(\boldsymbol{x}_r,\phi_r)}$ is a continuous function on Θ (cf. [3]).

Let φ be a continuous concave function on Θ. Clearly $\phi_{(\boldsymbol{x}_1,\varphi(\boldsymbol{x}_1)),\ldots,(\boldsymbol{x}_r,\varphi(\boldsymbol{x}_r))} \leq \varphi$. Moreover, for a positive number ϵ, if we add sufficiently many points $\boldsymbol{x}_{r+1}, \ldots, \boldsymbol{x}_m \in \Theta$ to $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$, then

$$\varphi - \epsilon \leq \phi_{(\boldsymbol{x}_1,\varphi(\boldsymbol{x}_1)),\ldots,(\boldsymbol{x}_r,\varphi(\boldsymbol{x}_r)),(\boldsymbol{x}_{r+1},\varphi(\boldsymbol{x}_{r+1})),\ldots,(\boldsymbol{x}_m,\varphi(\boldsymbol{x}_m))} \leq \varphi.$$

From now on, we use the same notation as in Section 1. We assume that $\overline{D}_{\boldsymbol{a}}$ is big.

Claim 6.1. *For a given positive number ϵ, we can find rational interior points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ of $\Theta_{\boldsymbol{a}}$, that is, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in \mathrm{Int}(\Theta_{\boldsymbol{a}}) \cap \mathbb{Q}^n$ such that*

$$\frac{(n+1)!}{2} \int_{\Theta} \phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x}) d\boldsymbol{x} > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon,$$

where $\Theta = \mathrm{Conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$.

Proof. First of all, we can find $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r'} \in \mathrm{Int}(\Theta_{\boldsymbol{a}}) \cap \mathbb{Q}^n$ such that

$$\frac{(n+1)!}{2} \int_{\Theta} \varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}) d\boldsymbol{x} > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon,$$

where $\Theta = \mathrm{Conv}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r'}\}$. Thus, adding more points $\boldsymbol{x}_{r'+1}, \ldots, \boldsymbol{x}_r \in \Theta \cap \mathbb{Q}^n$ to $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{r'}\}$, we have

$$\frac{(n+1)!}{2} \int_{\Theta} \phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x}) d\boldsymbol{x} > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon.$$

□

We choose a sufficiently small positive number δ such that

(a) $\Theta \subseteq \Theta_{e^{-\delta}\boldsymbol{a}}$ and

(b) $\dfrac{(n+1)!}{2}\displaystyle\int_{\Theta}\phi_{(\boldsymbol{x}_1,\varphi_{e^{-\delta}\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{e^{-\delta}\boldsymbol{a}}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})d\boldsymbol{x} > \widehat{\operatorname{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon.$

We set $\boldsymbol{a}' = e^{-\delta}\boldsymbol{a}$. By virtue of [9, Theorem3.2.3], we can find positive integer l_0 such that

(c) $\log\operatorname{dist}(H^0(lH_0)\otimes\mathbb{C}; l_0 g_{\boldsymbol{a}'}) \leq l_0\delta$ and
(d) $l_0\boldsymbol{x}_1, \ldots, l_0\boldsymbol{x}_r \in \mathbb{Z}^n_{\geq 0}$.

Let us consider the following $\mathbb{Z}$-module:

$$V := \bigoplus_{i=1}^{r}\mathbb{Z}z^{l_0\boldsymbol{x}_i} \subseteq H^0(\mathbb{P}^n_{\mathbb{Z}}, l_0H_0).$$

Then we have a birational morphisms $\mu : Y \to \mathbb{P}^n_{\mathbb{Z}}$ of projective, generically smooth and normal arithmetic varieties such that the image of

$$V\otimes_{\mathbb{Z}}\mathcal{O}_Y \to \mathcal{O}_Y(\mu^*(l_0H_0))$$

is invertible, that is, there is an effective Cartier divisor F on Y such that

$$V\otimes_{\mathbb{Z}}\mathcal{O}_Y \to \mathcal{O}_Y(\mu^*(l_0H_0)-F)$$

is surjective. Here we set

$$\begin{cases} Q := \mu^*(l_0H_0) - F, \\ g_F := \mu^*\left(-\log\operatorname{dist}(V\otimes\mathbb{C}; l_0g_{\boldsymbol{a}'}) + l_0\delta\right), \\ g_Q := \mu^*\left(l_0g_{\boldsymbol{a}'} + \log\operatorname{dist}(V\otimes\mathbb{C}; l_0g_{\boldsymbol{a}'})\right). \end{cases}$$

Claim 6.2. (i) $g_Q + g_F = \mu^*(l_0g_{\boldsymbol{a}})$.
(ii) *g_Q is a Q-Green function of $(C^\infty\cap\operatorname{PSH})$-type and $\overline{Q} := (Q, g_Q)$ is nef.*
(iii) *g_F is an F-Green function of C^∞-type and $g_F \geq 0$.*
(iv) *If we set $\overline{P} = (P, g_P) = (1/l_0)\overline{Q}$, then, for $\boldsymbol{e}\in l\Theta\cap\mathbb{Z}^n$, $\mu^*(z^{\boldsymbol{e}})\in H^0(lP)$ and*

$$|\mu^*(z^{\boldsymbol{e}})|^2_{lg_P} \leq \exp\left(-l\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{e}/l)\right).$$

Proof. (i) is obvious. (ii) is a consequence of Lemma 6.3 below. The first assertion of (iii) follows from (i) and (ii), and the second follows from (c).

(iv) Let us consider arbitrary $\lambda_1,\ldots,\lambda_r\in\mathbb{R}$ such that $\boldsymbol{e}/l = \lambda_1\boldsymbol{x}_1+\cdots+\lambda_r\boldsymbol{x}_r$ and $\lambda_1+\cdots+\lambda_r = 1$. Then, since $Q + (\mu^*(z^{l_0\boldsymbol{x}_i}))\geq 0$ for all i,

$$\begin{aligned} lP + (\mu^*(z^{\boldsymbol{e}})) &= (l/l_0)Q + \sum_{i=1}^{r}\lambda_i(l/l_0)(\mu^*(z^{l_0\boldsymbol{x}_i})) \\ &= \sum_{i=1}^{r}\lambda_i(l/l_0)\left(Q + (\mu^*(z^{l_0\boldsymbol{x}_i}))\right) \geq 0, \end{aligned}$$

and hence $\mu^*(z^{\boldsymbol{e}}) \in H^0(lP)$. Moreover, by using [9, Proposition 3.2.1] and Proposition 1.3,

$$\begin{aligned}
|\mu^*(z^{\boldsymbol{e}})|^2_{lg_P} &= |\mu^*(z^{\boldsymbol{e}})|^2 \exp(-(l/l_0)g_Q) \\
&= \prod_{i=1}^r \left(|\mu^*(z^{l_0\boldsymbol{x}_i})|^2\right)^{\lambda_i(l/l_0)} \frac{\exp(-l\mu^*(g_{\boldsymbol{a}'}))}{\mu^*(\operatorname{dist}(V\otimes\mathbb{C}; l_0 g_{\boldsymbol{a}'}))^{l/l_0}} \\
&= \prod_{i=1}^r \mu^*\left(\frac{|z^{l_0\boldsymbol{x}_i}|^2_{l_0 g_{\boldsymbol{a}'}}}{\operatorname{dist}(V\otimes\mathbb{C}; l_0 g_{\boldsymbol{a}'})}\right)^{\lambda_i(l/l_0)} \leq \prod_{i=1}^r \left(\|z^{l_0\boldsymbol{x}_i}\|^2_{l_0 g_{\boldsymbol{a}'}}\right)^{\lambda_i(l/l_0)} \\
&= \prod_{i=1}^r \exp(-l_0\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_i))^{\lambda_i(l/l_0)} = \exp\left(-l\sum_{i=1}^r \lambda_i\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_i)\right).
\end{aligned}$$

Thus (iv) follows. □

Lemma 6.3. *Let $\mu : Y \to X$ be a birational morphism of projective, generically smooth and normal arithmetic varieties. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X and S a subset of $\hat{H}^0(X, \overline{D})$. We assume that there is an effective $\mathbb{R}$-divisor E on Y with the following properties:*

(1) $\mu^*(D) - E \in \operatorname{Div}(Y)$*, that is,* $\mu^*(D) - E$ *is a Cartier divisor.*
(2) $\mu^*(s) \in H^0(Y, \mu^*(D) - E)$ *for all* $s \in S$ *and*

$$\bigcap_{s\in S} \operatorname{Supp}(\mu^*(D) - E + (\mu^*(s))) = \emptyset.$$

We set

$$M := \mu^*(D) - E \quad \textit{and} \quad g_M := \mu^*(g + \log\operatorname{dist}(\langle S\rangle_{\mathbb{C}}; g)).$$

Then g_M is an M-Green function of $(C^\infty \cap \mathrm{PSH})$-type and (M, g_M) is nef.

Proof. Let $e_1, \ldots, e_N$ be an orthonormal basis of $\langle S\rangle_{\mathbb{C}}$ with respect to $\langle\ ,\ \rangle_g$. We fix $y \in Y(\mathbb{C})$. Let f be a local equation of $\mu^*(D) - E$ around y. We set $s_j = \mu^*(e_j)f$ for $j = 1, \ldots, N$. Then $s_1, \ldots, s_N$ are holomorphic around y and $s_j(y) \neq 0$ for some j. On the other hand,

$$g_M = \log\left(\sum_{j=1}^N |\mu^*(e_j)|^2\right) = -\log|f|^2 + \log\left(\sum_{j=1}^N |s_j|^2\right)$$

around y. Thus g_M is an M-Green function of $(C^\infty \cap \mathrm{PSH})$-type. By virtue of [9, Proposition 3.1], we have

$$|s|^2_g \leq \langle s, s\rangle_g \operatorname{dist}(\langle S\rangle_{\mathbb{C}}; g) \leq \operatorname{dist}(\langle S\rangle_{\mathbb{C}}; g),$$

which yields $\mu^*(s) \in \hat{H}^0(Y, \overline{M})$ for all $s \in S$. Let C be a 1-dimensional closed integral subscheme on Y. Then there is $s \in S$ such that $C \not\subseteq \operatorname{Supp}(M + (\mu^*(s)))$. Thus $\widehat{\deg}(\,(M, g_M)|_C) \geq 0$. □

Finally let us see that $\widehat{\operatorname{vol}}(\overline{P}) > \widehat{\operatorname{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon$. We fix an F_∞-invariant volume form Φ on Y with $\int_{Y(\mathbb{C})} \Phi = 1$. Using Φ and lg_P, we can give the inner product $\langle\ ,\ \rangle_{lg_P}$ on $H^0(lP)$. Then, by (iv) in the above claim,

$$\langle \mu^*(z^{\boldsymbol{e}}), \mu^*(z^{\boldsymbol{e}})\rangle_{lg_P} \leq \exp\left(-l\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{e}/l)\right).$$

Here we consider positive definite symmetric real matrices $A_l = (a_{\boldsymbol{e},\boldsymbol{e}'})_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n}$ and $A'_l = (a'_{\boldsymbol{e},\boldsymbol{e}'})_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n}$ given by

$$a_{\boldsymbol{e},\boldsymbol{e}'} = \langle \mu^*(z^{\boldsymbol{e}}), \mu^*(z^{\boldsymbol{e}'})\rangle_{lg_P}$$

and

$$a'_{\boldsymbol{e},\boldsymbol{e}'} = \begin{cases} \exp\left(-l\phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{e}/l)\right) & \text{if } \boldsymbol{e} = \boldsymbol{e}', \\ \langle \mu^*(z^{\boldsymbol{e}}), \mu^*(z^{\boldsymbol{e}'})\rangle_{lg_P} & \text{if } \boldsymbol{e} \neq \boldsymbol{e}'. \end{cases}$$

Then, since

$$\sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \leq \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a'_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'},$$

we have

$$\begin{aligned} \#\hat{H}^0_{L^2}(l\overline{P}) &\geq \#\left\{(x_{\boldsymbol{e}}) \in \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n} \;\middle|\; \textstyle\sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \leq 1\right\} \\ &\geq \#\left\{(x_{\boldsymbol{e}}) \in \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n} \;\middle|\; \textstyle\sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a'_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \leq 1\right\}. \end{aligned}$$

On the other hand, by Lemma 2.2,

$$\begin{aligned} \liminf_{l\to\infty} & \frac{\log\#\left\{(x_{\boldsymbol{e}}) \in \mathbb{Z}^{l\Theta\cap\mathbb{Z}^n} \;\middle|\; \sum_{\boldsymbol{e},\boldsymbol{e}' \in l\Theta\cap\mathbb{Z}^n} a'_{\boldsymbol{e},\boldsymbol{e}'} x_{\boldsymbol{e}} x_{\boldsymbol{e}'} \leq 1\right\}}{l^{n+1}/(n+1)!} \\ & \geq \frac{(n+1)!}{2}\int_{\Theta} \phi_{(\boldsymbol{x}_1,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_1)),\ldots,(\boldsymbol{x}_r,\varphi_{\boldsymbol{a}'}(\widetilde{\boldsymbol{x}}_r))}(\boldsymbol{x})d\boldsymbol{x}, \end{aligned}$$

and hence $\widehat{\mathrm{vol}}(\overline{P}) > \widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) - \epsilon$ by Lemma 2.1 and (b).

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